On the reflection and transmission of Alfvén waves

Jingtao Min

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Abstract

Alfvén waves are magnetohydrodynamic (MHD) waves with Lorentz force as restoring force. Since initial proposition by Hannes Alfvén in 1950s, this type of waves have been recognized as an important component in plasma physics, astrophysics, as well as geophysics. They are considered to be associated with sunspots, differential rotation of different layers of planetary bodies, etc.

The behaviour of Alfvén waves at the medium boundary has probably already been studied by Alfvén in early publications, and has been more extensively studied by Ferraro (1954). Ferraro considers the interface between two diffusionless (inviscid as well as infinitely conductive) fluid media, in which scenario the closed-form travelling wave solutions for the reflected and refracted waves can be established. Most recently, Schaeffer, Jault, et al. (2012), Schaeffer and Jault (2016) studied the reflection behaviour at a solid wall, where a Hartmann boundary layer is necessary to satisfy the boundary conditions. Their model, however, is purely 1-dimensional, and does not incorporate any angular dependency of the problem. In the context of planetary and geophysical fluids, it is of interest to understand the general reflection and refraction properties of Alfvén waves, and due to the spherical geometry of celestial and planetary bodies, how Alfvén waves behave at boundaries at different latitudes, where they would most probably have diverse incidence directions. This article comprehensively reviews the techniques involved in analyzing the reflection and transmission properties of Alfvén waves, and aims to establish general solutions to this problem, including but not limited to the reflection and transmission with varying incidence orientations.

1 Alfvén waves

Alfvén waves are magnetohydrodynamic (MHD) waves with Lorentz force (or the magnetic tension) as restoring force. In this section I review some of the key aspects of this type of ways, including the equations, dispersion relations, phase and group velocities, etc.

1.1 Governing equations

I start by stating the governing equations of the system. Here it is already assumed that the medium is both homogeneous and incompressible, so that

$$\nabla \cdot \mathbf{u} = 0, \quad \rho \equiv Cst. \tag{1}$$

which already filters out all acoustic waves and stratification / arbitrary heterogeneity of the setup. Ferraro (1954) has a short section on Alfvén waves in stratified atmosphere, where the wave solution takes the form of Bessel functions, and is shown to be damped as it enters stratification region. This might be important in stars, planetary atmosphere, and might also become important if similar stratification occurs in Earth's core. When the incompressibility assumption is dropped, one obtains *magneto-acoustic* (*magnetosonic*) waves, which are hybrid between MHD waves and acoustic waves, with wave velocities in between Alfvén waves and acoustic waves. When both the fluctuation in density and the volumetric strain rate are comparatively negligible ([when should they be considered negligible?]), the assumptions should still work, and it should suffice in studying the reflection and transmission near the boundary.

I start from Navier-Stokes equation and induction equation in a homogeneous medium,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \frac{P}{\rho} + \frac{1}{\rho \mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{u}, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \tag{3}$$

The total fields ($\mathbf{u}$, $\mathbf{B}$) are decomposed into background fields ($\mathbf{U}_0$, $\mathbf{B}_0$) and perturbation fields ($\mathbf{u}$, $\mathbf{b}$). We take the background velocity field to be zero ($\mathbf{U}_0 = \mathbf{0}$), so that no advection occurs. The magnetic background field $\mathbf{B}_0$ is a constant in both space and time, representing a time-invariant uniform field. This approximation should hold as long as both the characteristic time scale and the length scale of $\mathbf{B}_0$ are much larger than those of the perturbed fields (in fact, the ratio should be greater than $|\mathbf{B}_0|/|\mathbf{b}|$).

To linearize the set of equations, the perturbation magnetic field is considered much smaller than the background field ($|\mathbf{b}| \ll |\mathbf{B}_0|$), and the perturbation velocity field is at the same order of magnitude as magnetic field, in the sense that $|\mathbf{u}| \sim |\mathbf{b}|/\sqrt{\rho\mu_0}$. Collecting only the first order terms, we obtain the linearized system

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= -\nabla \frac{P}{\rho} + \frac{1}{\rho \mu_0} (\nabla \times \mathbf{b}) \times \mathbf{B}_0 + \nu \nabla^2 \mathbf{u}, \\ \frac{\partial \mathbf{b}}{\partial t} &= \nabla \times (\mathbf{u} \times \mathbf{B}_0) + \eta \nabla^2 \mathbf{b}. \end{aligned}$$

which can be further rearranged into a more symmetric form using vector identities

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{\rho \mu_0} \mathbf{B}_0 \cdot \nabla \mathbf{b} + \nu \nabla^2 \mathbf{u} - \nabla P'_{\text{eff}}, \tag{4}$$

$$\frac{\partial \mathbf{b}}{\partial t} = \mathbf{B}_0 \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{b}. \tag{5}$$

where the effective pressure is a sum of mechanical pressure and the magnetic pressure, divided by the density of the medium

$$P'_{\text{eff}} = \frac{P}{\rho} + \frac{\mathbf{B}_0 \cdot \mathbf{b}}{\rho \mu_0}.$$

The total magnetic pressure is of course $(\mathbf{B_0} + \mathbf{b})^2/2\rho\mu_0$, but given the previous assumption that $\mathbf{B}_0$ is constant vector, and only the first-order terms are retained, using $(\mathbf{B}_0 + \mathbf{b})^2$ in the numerator would be equivalent to stating $2\mathbf{B}_0 \cdot \mathbf{b}$, which is what has been obtained directly from $(\nabla \times \mathbf{b}) \times \mathbf{B}_0$.

1.2 Dispersion relation of the ideal system

Neglecting the pressure gradient, and neglecting the viscous as well as magnetic diffusion, the equations can be combined into a second-order wave equation, in the form

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{(\mathbf{B}_0 \cdot \nabla)^2}{\rho \mu_0} \mathbf{u}, \qquad \frac{\partial^2 \mathbf{b}}{\partial t^2} = \frac{(\mathbf{B}_0 \cdot \nabla)^2}{\rho \mu_0} \mathbf{b}, \tag{6}$$

which, with a plane wave ansatz of any sort (equivalently converting the equation into frequency-wavenumber domain), immediately yields the dispersion relation for Alfvén waves in diffusionless medium

$$\omega^2 = \frac{1}{\rho \mu_0} (\mathbf{k} \cdot \mathbf{B}_0)^2 = \frac{B_0^2}{\rho \mu_0} \left(\mathbf{k} \cdot \widehat{\mathbf{B}}_0\right)^2 = V_A^2 \left(\mathbf{k} \cdot \widehat{\mathbf{B}}_0\right)^2. \tag{7}$$

where $\widehat{\mathbf{B}}_0$ is the unit vector in the direction of $\mathbf{B}_0$, and $V_A = B_0/\sqrt{\rho\mu_0}$ is the Alfvén wave velocity, as will become clearer in the next section. For the plane wave ansatz used in this article, please refer to eq.(9) and the related texts in the remark box. Since reversing the sign on ω and $\mathbf{k}$ simultaneously would yield the same physical solution (taking the real part of the complex wave yields the exact same expression, see remark box that follows), when describing the plane waves I shall take the convention that $\omega > 0$. Under this convention the dispersion relation can be further written as

$$\omega = \pm V_A \left(\mathbf{k} \cdot \widehat{\mathbf{B}}_0\right) = \begin{cases} V_A \left(\mathbf{k} \cdot \widehat{\mathbf{B}}_0\right), & \mathbf{k} \cdot \widehat{\mathbf{B}}_0 > 0 \\ -V_A \left(\mathbf{k} \cdot \widehat{\mathbf{B}}_0\right), & \mathbf{k} \cdot \widehat{\mathbf{B}}_0 < 0 \end{cases} \tag{8}$$

The two solutions for ω correspond to two propagation directions of the Alfvén wave. As will also become clear later, Alfvén waves can either propagate in or opposite the direction of the background magnetic field, which corresponds to the obtained two solutions.

It is already readily seen that Alfvén wave is an anisotropic wave. The isotropy is broken due to the fact that background magnetic field is the essential cornerstone for providing the magnetic tension, and the orientation of the magnetic field has a special status. It will also be seen that the orientation of the background magnetic field greatly complicates the reflection-refraction problem, compared to the isotropic waves, such as elastic waves, acoustic waves (seismic waves) and light waves in isotropic medium. The anisotropy dictates that, for a given temporal frequency ω,

$$\left|\mathbf{k} \cdot \widehat{\mathbf{B}}_0\right| = \frac{\omega}{V_A}$$

meaning the wave vector has a fixed projection length on the background field.

Plane wave ansatz for Alfvén waves

In this article I shall use the following convention for plane wave

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 \exp\left\{i\left(\omega t - \mathbf{k} \cdot \mathbf{r}\right)\right\}, \tag{9}$$

and the conventions for respective Fourier transforms follow. Some intermediate steps and results in this article will be different by a sign compared to the alternative ansatz $\exp\{i(\omega t + \mathbf{k} \cdot \mathbf{r})\}$, for instance the dispersion relation as shown in eq.(8). In the other convention, $\mathbf{k}$ is opposite the direction in which the phase propagates, and $\omega = V_A(\mathbf{k} \cdot \widehat{\mathbf{B}}_0)$ would represent a wave travelling in the opposite direction of $\mathbf{B}_0$.

In the ideal case without diffusion, it can be easily shown that the perturbation velocity field $\mathbf{u}$ and the magnetic field $\mathbf{b}$ are proportional to one another. To this end, we can construct a plane wave solution for the perturbed fields

$$\mathbf{b} = \mathbf{b}_0 \exp\{i(\omega t - \mathbf{k} \cdot \mathbf{r})\}, \quad \mathbf{u} = \mathbf{u}_0 \exp\{i(\omega t - \mathbf{k} \cdot \mathbf{r})\}.$$

The two waves share the same phase argument since their phases need to match in the coupled system of equations. Substituting the expression into the first-order ideal equation yields

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\mathbf{B}_0 \cdot \nabla}{\rho \mu_0}\mathbf{b}, \quad \Longrightarrow \quad i\omega \mathbf{u}_0 = -i\frac{\mathbf{B}_0 \cdot \mathbf{k}}{\rho \mu_0}\mathbf{b}_0 \quad \Longrightarrow \quad \mathbf{u}_0 = -\frac{V_A(\mathbf{k} \cdot \widehat{\mathbf{B}}_0)}{\omega \sqrt{\rho \mu_0}}\mathbf{b}_0.$$

Taking into account the dispersion relation, we have

$$\mathbf{u}_0 = \begin{cases} -\dfrac{\mathbf{b}_0}{\sqrt{\rho \mu_0}}, & \mathbf{k} \cdot \widehat{\mathbf{B}}_0 > 0, \\ \dfrac{\mathbf{b}_0}{\sqrt{\rho \mu_0}}, & \mathbf{k} \cdot \widehat{\mathbf{B}}_0 < 0. \end{cases} \tag{10}$$

Therefore, the perturbed magnetic field and the velocity field are opposite one another (or have a π phase shift) when the wave is propagating in the same direction of the background magnetic field, and the perturbed fields are completely in-phase when the wave is propagating in the opposite direction of $\mathbf{B}_0$. Either way, the magnetic field and the velocity field in Alfvén waves share the same polarization.

1.3 Dispersion relation of the diffusive system

With viscous and magnetic diffusion, the system of equations cannot be easily combined into one wave equation. Instead, the system of equations can be converted directly into frequency-wavenumber domain

$$\begin{aligned} i\omega\mathbf{u} &= -i\frac{\mathbf{B}_0\cdot\mathbf{k}}{\rho\mu_0}\mathbf{b} - \nu k^2\mathbf{u}, \\ i\omega\mathbf{b} &= -i(\mathbf{B}_0\cdot\mathbf{k})\mathbf{u} - \eta k^2\mathbf{b}, \end{aligned} \tag{11}$$

which can be rearranged into the linear system

$$\begin{aligned} (i\omega + \nu k^2)\mathbf{u} + i\frac{\mathbf{B}_0\cdot\mathbf{k}}{\rho\mu_0}\mathbf{b} &= \mathbf{0}, \\ i(\mathbf{B}_0\cdot\mathbf{k})\mathbf{u} + (i\omega + \eta k^2)\mathbf{b} &= \mathbf{0}. \end{aligned} \tag{12}$$

Naturally, for the system to have nontrivial solutions, the necessary condition is

$$\det\begin{pmatrix} i\omega + \nu k^2 & i\frac{\mathbf{B}_0\cdot\mathbf{k}}{\rho\mu_0} \\ i(\mathbf{B}_0\cdot\mathbf{k}) & i\omega + \eta k^2 \end{pmatrix} = -\omega^2 + i(\nu+\eta)k^2\omega + \frac{(\mathbf{B}_0\cdot\mathbf{k})^2}{\rho\mu_0} + \nu\eta k^4 = 0$$

which is the dispersion relation for the diffusive system. The relation can be rearranged as a biquadratic polynomial equation of k,

$$\nu\eta k^4 + \left(V_A^2\cos^2\gamma + i\omega(\nu+\eta)\right)k^2 - \omega^2 = 0 \tag{13}$$

where I have used $\gamma = \langle \mathbf{B}_0, \mathbf{k}\rangle$, and the defined Alfvén wave velocity $V_A = B_0/\sqrt{\rho\mu_0}$. The roots of this equation give the spatial branch of the dispersion relations

$$k^2 = -\frac{V_A^2\cos^2\gamma}{2\nu\eta}\left(1 + i\frac{\omega(\nu+\eta)}{V_A^2\cos^2\gamma}\right)\left[1 \pm \sqrt{1 + \frac{4\omega^2\nu\eta}{V_A^4\cos^4\gamma\left(1 + \frac{i\omega(\nu+\eta)}{V_A^2\cos^2\gamma}\right)^2}}\right]. \tag{14}$$

The repetitive terms can be greatly simplified by introducing the notation

$$\mathrm{S}_\omega = \frac{2V_A^2}{\omega(\nu+\eta)} \tag{15}$$

and the spatial branch of the dispersion relation can be rewritten as

$$k^2 = -\frac{V_A^2}{2\nu\eta}\left(\cos^2\gamma + i2\mathrm{S}_\omega^{-1}\right)\left[1 \pm \sqrt{1 + \frac{4\omega^2\nu\eta}{V_A^4\left(\cos^2\gamma + i2\mathrm{S}_\omega^{-1}\right)^2}}\right] \tag{16}$$

The notation S_ω defined as such represents the ratio between the diffusion time scale and the Alfvén wave time scale, and is called the *Lundquist number*. In its general form, without specifying the length scale of interest, the Lundquist number is written as

$$\mathrm{S} = \frac{\tau_\alpha}{\tau_A} = \frac{L^2/\alpha}{L/V_A} = \frac{V_A L}{\alpha}. \tag{17}$$

Choosing specific diffusion mechanism (specifying α) and specific time scale (specifying L) yields a variety of variants of Lundquist number. In this case, we see that the diffusion mechanism of interest is the combined effect of viscous diffusion and magnetic diffusion; the length scale of interest is determined by the Alfvén wavelength at specified frequency, i.e.

$$\alpha \sim \frac{\nu+\eta}{2}, \quad L \sim \frac{V_A}{\omega} \quad \Longrightarrow \quad \mathrm{S}_\omega = \frac{V_A^2/\omega}{(\nu+\eta)/2} = \frac{2V_A^2}{\omega(\nu+\eta)}.$$

When Lundquist number $\mathrm{S} \gg 1$, Alfvén wave time scale is much smaller than that of diffusion time scale; this means the damping of Alfvén waves at the specific length scale is small, and the propagation of Alfvén waves is allowed in the system. When $\mathrm{S} \ll 1$, diffusion time scale is much smaller, and diffusion process dominates the system at given length scale, prohibiting the effective propagation of Alfvén waves. This can be readily seen from the dispersion relation, as follows.

At $\mathrm{S}_\omega \ll 1$, the term S_ω^{-1} always dominates over other terms, reducing eq.(16) into the form

$$k^2 \approx -i\frac{V_A^2}{2\nu\eta}2\mathrm{S}_\omega^{-1}\left(1 \pm \sqrt{1-\frac{\omega^2\nu\eta}{V_A^4}\mathrm{S}_\omega^2}\right) = -i\frac{\omega}{2\nu\eta}\left(\nu+\eta\pm|\nu-\eta|\right)$$

which gives the ultimate solutions

$$k_1^2 \approx -i\frac{\omega}{\min(\nu,\eta)}, \quad k_1 = \pm\frac{1-i}{\sqrt{2}}\sqrt{\frac{\omega}{\min(\nu,\eta)}}, \tag{18}$$

$$k_2^2 \approx -i\frac{\omega}{\max(\nu,\eta)}, \quad k_2 = \pm\frac{1-i}{\sqrt{2}}\sqrt{\frac{\omega}{\max(\nu,\eta)}}. \tag{19}$$

These solutions correspond to damped oscillations, which decays in the propagating direction. The characteristic decaying length is given by $\bar{\lambda}_1 = \sqrt{2\min(\nu,\eta)/\omega}$ and $\bar{\lambda}_2 = \sqrt{2\max(\nu,\eta)/\omega}$, respectively. The $\sqrt{\omega/\alpha}$ scaling for k, the characteristic wavelength scaling with $\sqrt{\alpha/\omega}$ and the feature of damped oscillation reveal that these solutions correspond to *Stokes*-type boundary layers. The oscillation and damping is simply governed by the diffusion process, but not the magnetic tension.

At $\mathrm{S}_\omega \gg 1$, or $V_A^2 \gg \omega(\nu+\eta) \geq 2\omega\sqrt{\nu\eta}$, we expect to recover the regime where magnetic tension is important in the system, producing the propagation of Alfvén waves. Given some value of γ so that $\cos\gamma \sim 1$ is at some finite magnitude, the inverse Lundquist number S_ω^{-1} will always be small compared to $\cos^2\gamma$. To see the effect of diffusion in the system we can keep S_ω^{-1} in eq.(16) to its leading order,

$$\begin{aligned} k^2 &= -\frac{V_A^2\cos^2\gamma}{2\nu\eta}\left(1+i\frac{2}{\mathrm{S}_\omega\cos^2\gamma}\right)\left[1 \pm \sqrt{1+\frac{4\omega^2\nu\eta}{V_A^4\cos^4\gamma}\left(1+i\frac{2}{\mathrm{S}_\omega\cos^2\gamma}\right)^{-2}}\right] \\ &\approx -\frac{V_A^2\cos^2\gamma}{2\nu\eta}\left(1+i\frac{2}{\mathrm{S}_\omega\cos^2\gamma}\right)\left[1 \pm \left(1+\frac{2\omega^2\nu\eta}{V_A^4\cos^4\gamma}\left(1-i\frac{4}{\mathrm{S}_\omega\cos^2\gamma}\right)\right)\right] \end{aligned}$$

which gives two ultimate solutions

$$k_1^2 \approx -\frac{V_A^2}{\nu\eta}\left(\cos^2\gamma + i2\mathrm{S}_\omega^{-1}\right), \quad k_1 \approx \pm\frac{V_A\cos\gamma}{\sqrt{\nu\eta}}\left(-\frac{1}{\mathrm{S}_\omega\cos^2\gamma}+i\right), \tag{20}$$

$$k_2^2 \approx \frac{\omega^2}{V_A^2\cos^2\gamma}\left(1-i\frac{2}{\mathrm{S}_\omega\cos^2\gamma}\right), \quad k_2 \approx \pm\frac{\omega}{V_A\cos\gamma}\left(1-i\frac{1}{\mathrm{S}_\omega\cos^2\gamma}\right). \tag{21}$$

The first solution k_1 has a dominant imaginary part. In the case where $\mathrm{S}_\omega^{-1} \ll 1$ is negligible this can be simply written as $k_1 \approx \pm iV_A\cos\gamma/\sqrt{\nu\eta} = \pm i/\delta$. This correspond to a *Hartmann boundary layer*. Properties of this layer is listed below.

Hartmann boundary layer

The thickness, or the characteristic length scale over which the wave decays in a Hartmann layer, is given by

$$\delta_{\mathrm{BL}} = \frac{\sqrt{\nu\eta}}{V_A|\cos\gamma|} = \frac{\sqrt{\rho\mu_0\nu\eta}}{B_0|\cos\gamma|} = \frac{1}{B_\parallel}\sqrt{\frac{\rho\nu}{\sigma}}. \tag{22}$$

Here $B_\parallel = |B_0\cdot\hat{\mathbf{k}}|$ is the magnetic field strength in line with the wave vector. The boundary layer thickness scales with $\sqrt{\nu\eta}/V_A$, and hence goes to zero when $\nu,\eta \to 0$. Nevertheless, an infinitely small Hartmann layer might be able to accommodate finite velocity and magnetic field discontinuity at the boundary ([is this true?]), just like the free-slip boundary condition for inviscid fluid. The reason for that is the infinite conductivity assumption, which allows infinitely large current in the system, giving rise to magnetic field discontinuity. This boundary layer seems to play an important role in constructing solutions that satisfy continuity boundary conditions across the interface (Schaeffer, Jault, et al. 2012), even at high Lundquist numbers.

The thickness of the Hartmann boundary layer is another important length scale of the system. When variation occurs on a length scale comparable to or smaller than δ_{BL}, the viscous and magnetic diffusion wins over, and promotes boundary layer behaviour; when the length scale of variation is much larger than δ_{BL}, the magnetic tension is much more effective. This motivates the introduction of a dimensionless number, *Hartmann number*, which, supposedly, is the ratio of Lorentz force to viscous force. It can also be interpreted as the ratio of some characteristic length scale to the Hartmann layer thickness

$$\mathrm{Ha} = BL\sqrt{\frac{\sigma}{\rho\nu}} = \frac{L}{\delta_{\mathrm{BL}}}. \tag{23}$$

For Hartmann layer, the amplitudes of $\mathbf{u}$ and $\mathbf{b}$ follow a relation different from the travelling wave solution. Recall at high Lundquist number, $V_A^2 \gg \omega\eta$, the first-order equation gives

$$\mathbf{u} = \frac{i\mathbf{B}_0\cdot\mathbf{k}}{\rho\mu_0(i\omega+\nu k^2)}\mathbf{b} \approx \frac{\mp B_0\frac{V_A\cos^2\gamma}{\sqrt{\nu\eta}}}{\rho\mu_0\frac{V_A^2}{\eta}\cos^2\gamma}\mathbf{b} = \mp\sqrt{\frac{\eta}{\nu}}\frac{\mathbf{b}}{\sqrt{\rho\mu_0}} = \mp\mathrm{Pm}^{-1/2}\frac{\mathbf{b}}{\sqrt{\rho\mu_0}}$$

where $\mathrm{Pm} = \frac{\nu}{\eta}$ is the magnetic Prandtl number. Not surprisingly, the amplitude in such boundary layer is skewed towards the field with smaller diffusion.

The second solution k_2 reduces to the dispersion relation of Alfvén waves in diffusionless medium, i.e. $k_2 = \pm\omega/V_A\cos\gamma$, when S_ω^{-1} is dropped from the multiplier (eq.8). To first order, the role of non-negligible diffusion is to introduce damping with the coefficient $1/\mathrm{S}_\omega\cos^2\gamma = \omega(\nu+\eta)/2V_A^2\cos^2\gamma$, which mildly damps the Alfvén wave as it propagates.

At the mildly diffusive limit $\mathrm{S}_\omega \gg 1$, both solutions are anisotropic. For the Hartmann boundary layer solution, the thickness or spatial decay rate is constrained by the projection of magnetic field on the wave vector. For the travelling Alfvén wave solution, the wavenumber is also determined by the projection of the magnetic field on the wave vector. Conversely, it means the wave vectors in both solutions are only controlled in the direction of the background field.

1.4 Phase and group velocities

Phase velocity is the velocity at which the phase of a monochromatic wave travels. Given that a plane wave takes the form $\exp(i(\omega t - \mathbf{k}\cdot\mathbf{r})) = \exp(i\mathbf{k}\cdot(\omega t\hat{\mathbf{k}}/k - \mathbf{r}))$, the phase velocity is given by

$$\mathbf{c}_p = \frac{\omega}{k}\hat{\mathbf{k}}. \tag{24}$$

Apparently, the phase velocity always has the same direction as the wave vector. The wave vector for plane wave is exactly the indicator of phase propagation. When a collection of waves is present (in reality there is almost never standalone monochromatic plane wave, since that implies infinite energy), the velocity at which the wave packet near a frequency travels is different from the phase velocity. This is called the group velocity, and is given by

$$\mathbf{c}_g = \nabla_k\omega. \tag{25}$$

Since wave packets and wave groups are the carrier of information and energy, group velocity is considered to be the velocity at which information and energy propagates. Although this seems to be true in many cases, I argue that this cannot replace the energy argument. Group velocity is a mathematical property, a property that arises due to the mathematical form of wave ansätze; energy flux is a physical property, which does not seem to be strictly linked to the wave ansatz *a priori*.

Both phase velocity and the group velocity can be derived from the dispersion relations. The dispersion relation with finite S_ω is complicated, and would give rise to dispersive waves. The Alfvén wave in ideal, diffusionless medium, however, is simple. The phase velocity is given by

$$\mathbf{c}_p = \frac{\omega}{k}\hat{\mathbf{k}} = \begin{cases} V_A(\hat{\mathbf{k}}\cdot\widehat{\mathbf{B}}_0)\hat{\mathbf{k}} = V_A\cos\gamma\,\hat{\mathbf{k}} = \dfrac{\mathbf{B}_0\cdot\hat{\mathbf{k}}}{\sqrt{\rho\mu_0}}\hat{\mathbf{k}}, & \mathbf{k}\cdot\widehat{\mathbf{B}}_0 > 0 \\ -V_A(\hat{\mathbf{k}}\cdot\widehat{\mathbf{B}}_0)\hat{\mathbf{k}} = -V_A\cos\gamma\,\hat{\mathbf{k}} = -\dfrac{\mathbf{B}_0\cdot\hat{\mathbf{k}}}{\sqrt{\rho\mu_0}}\hat{\mathbf{k}}, & \mathbf{k}\cdot\widehat{\mathbf{B}}_0 < 0 \end{cases} \tag{26}$$

which can be uniformly written as

$$\mathbf{c}_p = V_A |\cos\gamma| \,\hat{\mathbf{k}}. \tag{27}$$

The magnitude of the phase velocity of Alfvén wave is fixed, as long as the orientation of the wave propagation is fixed. In this sense, Alfvén wave is diffusionless. Among all the direction of waves, the wave that propagates along the background field propagates the fastest.

For isotropic waves such as seismic waves and light waves in isotropic medium, not only are the waves dispresionless, but $\mathbf{c}_p \parallel \mathbf{c}_g$. This is not the case for anisotropic waves as Alfvén wave. While the phase can travel in any direction except normal to the background field, the group velocity is always aligned with the background field

$$\mathbf{c}_g = \nabla_k \omega = \begin{cases} V_A \widehat{\mathbf{B}}_0 = \dfrac{\mathbf{B}_0}{\sqrt{\rho\mu_0}}, & \mathbf{k}\cdot\widehat{\mathbf{B}}_0 > 0, \\ -V_A \widehat{\mathbf{B}}_0 = -\dfrac{\mathbf{B}_0}{\sqrt{\rho\mu_0}}, & \mathbf{k}\cdot\widehat{\mathbf{B}}_0 < 0. \end{cases} \tag{28}$$

For an Alfvén wave propagating (in the sense of $\mathbf{c}_p$ or $\mathbf{k}$) in the direction that forms an angle $\gamma < \pi/2$ with the magnetic field $\mathbf{B}_0$, i.e. downwind, the group velocity is *in* the direction of $\mathbf{B}_0$. For an Alfvén wave propagating upwind, the group velocity is *opposite* the direction of $\mathbf{B}_0$. Either way, the magnitude of group velocity is always given by the Alfvén wave velocity V_A.

1.5 Energy and energy flux

[Energy properties, I don't think I understand the energy perspective well enough.]

2 1-D Reflection at solid boundary

Reflection of Alfvén waves at fluid-solid boundaries in 1-D has been studied by Schaeffer, Jault, et al. (2012) for insulating solid boundaries, and Schaeffer and Jault (2016) for solid boundaries with a conductive layer. In both of these cases, the solution in the fluid is composed of (i) an incoming travelling Alfvén wave, (ii) a reflected travelling Alfvén wave, and (iii) a Hartmann layer. The solution is then complemented with boundary conditions on the fluid-solid boundary. In the presence of conductive layer in the solid wall, the solution in the fluid part is further paired with a purely electromagnetic wave solution in the conductive layer. The solution of the electromagnetic wave in the conductive or insulating solid in its general form is given in the following box.

Electromagnetic waves in the solid

As preliminary information, I collect here the solutions and dispersion relations of EM waves in the solid. To ensure that in the insulating limit, the equation does not degenerate into Possion's equation, where the energy flux is no longer present, I shall keep the displacement currents. Assuming negligible fluctuation in electric permittivity and magnetic permeability, the Maxwell equations in the electrically neutral solid interior take the form in the Fourier domain

$$\begin{aligned} &\nabla \cdot \mathbf{E} = 0 \\ &\nabla \times \mathbf{E} = -i\omega \mathbf{B} \\ &\nabla \cdot \mathbf{B} = 0 \\ &\nabla \times \mathbf{B} = \mu_0 \sigma \mathbf{E} + i\omega\mu_0\varepsilon_0 \mathbf{E} \end{aligned} \tag{29}$$

Recall that the light speed $c = 1/\sqrt{\mu_0\varepsilon_0}$, the magnetic diffusivity $\eta = 1/\mu_0\sigma$, and take the curl of the Ampere's law, we arrive at the (damped) wave equation of either $\mathbf{B}$ or $\mathbf{E}$

$$\nabla^2 \mathbf{B} - i\frac{\omega}{\eta}\mathbf{B} + \frac{\omega^2}{c^2}\mathbf{B} = 0 \tag{30}$$

where ω/η serves as a damping term to the equation. Transforming the equation further into the wavenumber domain, we obtain the dispersion relation

$$k^2 = -i\frac{\omega}{\eta} + \frac{\omega^2}{c^2} = -i\omega\mu_0\sigma + \omega^2\mu_0\varepsilon_0. \tag{31}$$

The contribution is clear-cut: conduction current contributes to the first term on the right-hand-side, while displacement current contributes to the latter term. In the limit of absolute insulator, all one needs to do is to remove the first term. The solutions are more explicitly given by

$$\begin{aligned} k &= \pm\frac{\omega}{c}\sqrt{\frac{1}{2}\left(1+\sqrt{1+\frac{c^4}{\omega^2\eta^2}}\right)} \mp i\frac{\omega}{c}\sqrt{\frac{1}{2}\left(-1+\sqrt{1+\frac{c^4}{\omega^2\eta^2}}\right)} \\ &= \pm\frac{\omega}{c}\sqrt{\frac{1}{2}\left(1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon_0^2}}\right)} \mp i\frac{\omega}{c}\sqrt{\frac{1}{2}\left(-1+\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon_0^2}}\right)}. \end{aligned} \tag{32}$$

We consider two limits. In the quasi-static limit, where $\sigma \gg \omega\varepsilon_0$, the conduction current dominates, leading to the dispersion relation

$$k = \pm\frac{1-i}{\sqrt{2}}\sqrt{\omega\sigma\mu_0} = \pm\frac{1-i}{\sqrt{2}}\sqrt{\frac{\omega}{\eta}} = \pm\frac{1-i}{\delta_s}. \tag{33}$$

This yields the so-called skin-depth, defined as $\delta_s = \sqrt{2/\omega\sigma\mu_0} = \sqrt{2\eta/\omega}$. The electromagnetic field decays at this length scale, and the resulting solution is a strongly damped oscillation, analogous to the Stokes boundary layer. At the other limit, $\sigma \ll \omega\varepsilon_0$, we recover the mildly diffusive travelling wave

$$k = \pm\frac{\omega}{c} \mp i\frac{\sigma}{2}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \pm\frac{\omega}{c} \mp i\frac{c}{2\eta}, \tag{34}$$

where the length scale on which the field decays $\sim \eta/c$ is much greater than the wavelength of the electromagnetic wave $\lambda \sim c/\omega$. In particular, if we take it to the insulating limit, i.e. $\sigma \to 0$ or $\eta \to +\infty$, we recover the electromagnetic wave in the absolutely insulating medium, equivalent to the propagation in vacuum $k = \omega/c$.

2.1 Normal incidence at insulating solid wall

This scenario is treated in Schaeffer, Jault, et al. (2012).

2.1.1 Problem setup

We consider the incidence of an Alfvén wave travelling in the z-direction at the fluid-solid interface, located at $z = 0$. A fluid of density ρ occupies the space $z < 0$, and an insulating solid occupies the space $z > 0$. The uniform magnetic field $\mathbf{B}_0$ is directed in the z-direction, i.e. $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$. The setup is representative of a plane Alfvén wave impinging normal to the interface, in a background field normal to the interface.

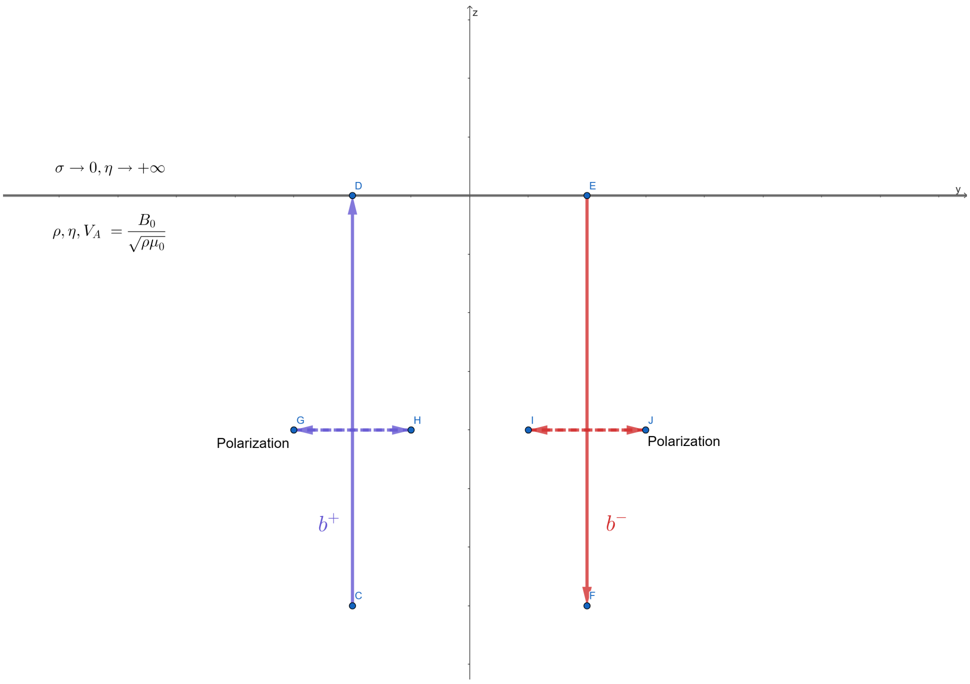

Figure 1: Setup of Alfvén wave normal incidence at insulating boundary

As one would imagine, the problem is purely 1-D, and can be described by 1-D version of the governing equations (see Schaeffer, Jault, et al. 2012). Here I proceed directly to stating the plane wave solutions in 1-D, simply by taking $\mathbf{k} = k\hat{\mathbf{z}}$ and $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ and picking the polarization in the y-direction. Naturally $\gamma = 0$ follows. According to the dispersion relation at high Lundquist number, we have the travelling wave solution

$$\mathbf{b}^+ = b^+\hat{\mathbf{y}} \exp\left\{i\omega\left(t - \frac{z}{V_A}\right)\right\}, \quad \mathbf{u}^+ = u^+\hat{\mathbf{y}} \exp\left\{i\omega\left(t - \frac{z}{V_A}\right)\right\} = -\frac{b^+\hat{\mathbf{y}}}{\sqrt{\rho\mu_0}} \exp\left\{i\omega\left(t - \frac{z}{V_A}\right)\right\} \tag{35}$$

$$\mathbf{b}^- = b^-\hat{\mathbf{y}} \exp\left\{i\omega\left(t + \frac{z}{V_A}\right)\right\}, \quad \mathbf{u}^- = u^-\hat{\mathbf{y}} \exp\left\{i\omega\left(t + \frac{z}{V_A}\right)\right\} = +\frac{b^-\hat{\mathbf{y}}}{\sqrt{\rho\mu_0}} \exp\left\{i\omega\left(t - \frac{z}{V_A}\right)\right\} \tag{36}$$

where $\mathbf{b}^+$ and $\mathbf{u}^+$ give the wave travelling in the positive z-direction, in the same direction as $\mathbf{B}_0$, and $\mathbf{b}^-$ and $\mathbf{u}^-$ give the wave travelling in the negative z-direction. In addition to that, we have the Hartmann boundary layer solution, which takes the form

$$\mathbf{b}^{\mathrm{BL}} = b^{\mathrm{BL}}\hat{\mathbf{y}} \exp\left\{i\omega t + \frac{V_A z}{\sqrt{\nu\eta}}\right\}, \quad \mathbf{u}^{\mathrm{BL}} = u^{\mathrm{BL}} \exp\left\{i\omega t + \frac{V_A z}{\sqrt{\nu\eta}}\right\} = -\frac{1}{\sqrt{\mathrm{Pm}}}\frac{\mathbf{b}^{\mathrm{BL}}}{\sqrt{\rho\mu_0}} \tag{37}$$

These are already solutions to the Alfvén wave equation in high Lundquist number approximation.

With magnetic and viscous diffusion both present, the system requires no-slip boundary condition for kinematics, and continuity of magnetic field for electromagnetic boundary conditions. The continuity of electric field is automatically satisfied given these two conditions. Within the insulating solid wall, the magnetic field is zero. This seems to be a non-trivial assumption, but may be justified in the 1-D case either by invoking finite energy, or by arguing that the toroidal field in the half space needs to be zero. [Is this really the case? Or is it actually necesary to invoke another wave in the insulating media and imposing continuity of electric field?] The consequence is that instead of constructing simultaneously the solution within the solid region, we can simply impose the homogeneous Dirichlet boundary conditions

$$u_y|_{z=0^-} = 0, \quad b_y|_{z=0^-} = 0. \tag{38}$$

These completes the system of equations.

2.1.2 Solutions for reflected wave and boundary layer

Substituting the ansatz

$$\begin{aligned} b_y &= b^+ \exp\left\{i\omega\left(t - \frac{z}{V_A}\right)\right\} + b^- \exp\left\{i\omega\left(t + \frac{z}{V_A}\right)\right\} + b^{\mathrm{BL}} \exp\left\{i\omega t + \frac{V_A z}{\sqrt{\nu\eta}}\right\}, \\ u_y &= -\frac{b^+}{\sqrt{\rho\mu_0}} \exp\left\{i\omega\left(t - \frac{z}{V_A}\right)\right\} + \frac{b^-}{\sqrt{\rho\mu_0}} \exp\left\{i\omega\left(t + \frac{z}{V_A}\right)\right\} - \frac{\mathrm{Pm}^{-\frac{1}{2}} b^{\mathrm{BL}}}{\sqrt{\rho\mu_0}} \exp\left\{i\omega t + \frac{V_A z}{\sqrt{\nu\eta}}\right\} \end{aligned} \tag{39}$$

into the two boundary conditions, we find that the phases of the terms naturally match (in 1-D there is no spatial depedency of the phase on Oxy plane, so phase-matching is a trivial condition). The amplitudes satisfy the equations

$$\begin{aligned} b^+ + b^- + b^{\mathrm{BL}} &= 0, \\ b^+ - b^- + \mathrm{Pm}^{-\frac{1}{2}} b^{\mathrm{BL}} &= 0, \end{aligned} \tag{40}$$

which yield the solutions

$$\frac{b^{\mathrm{BL}}}{b^+} = -\frac{2\sqrt{\mathrm{Pm}}}{1+\sqrt{\mathrm{Pm}}}, \qquad \frac{u^{\mathrm{BL}}}{u^+} = -\frac{2}{1+\sqrt{\mathrm{Pm}}}, \tag{41}$$

$$R_b = \frac{b^-}{b^+} = -\frac{1-\sqrt{\mathrm{Pm}}}{1+\sqrt{\mathrm{Pm}}}, \qquad R_u = \frac{u^-}{u^+} = \frac{1-\sqrt{\mathrm{Pm}}}{1+\sqrt{\mathrm{Pm}}}. \tag{42}$$

Eq.(42) gives the reflection coefficients for the magnetic field and the velocity field, and eq.(41) gives the amplitudes of the fields in the Hartmann layer. All of the aforementioned coefficients depend solely on the magnetic Prandtl number.

2.2 Normal incidence at conductive layer

This scenario is treated in Schaeffer and Jault (2016).

2.2.1 Problem setup

The setup of the problem is almost identical to the previous, except for a conductive layer of electrical conductivity σ_w and thickness h. This changes two things in the previous setup. First, there can be a purely electromagnetic field in the conductive solid wall, where the wavenumber of the electromagnetic wave can be directly taken from the previous box

$$k_w = \sqrt{\frac{\omega\mu_0\sigma_w}{2}}(1-i) = \sqrt{\frac{\omega}{2\eta_w}}(1-i) = \frac{1-i}{\delta_w}$$

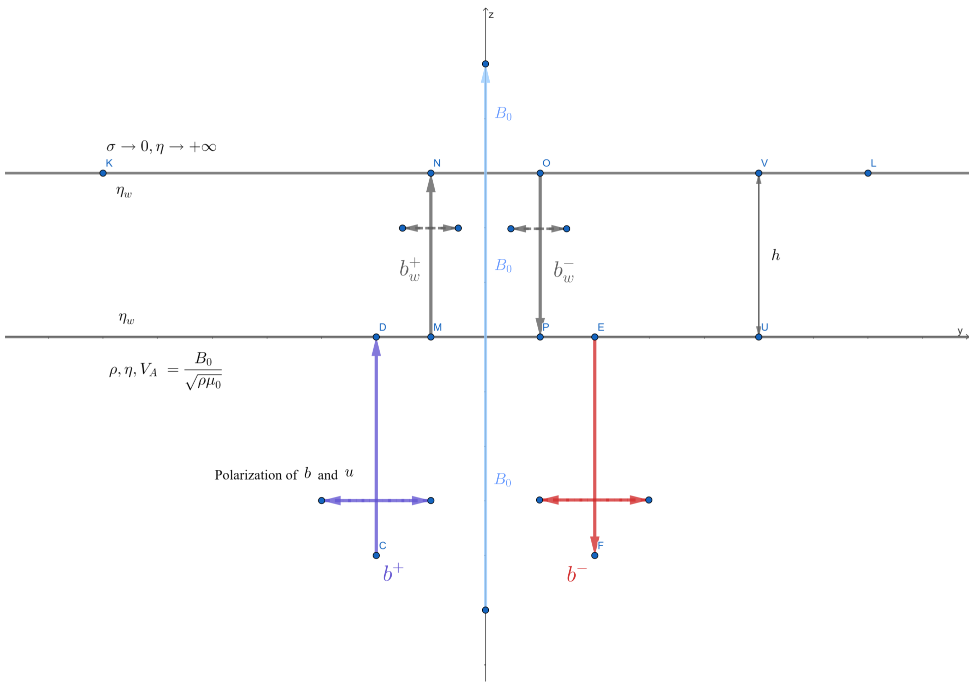

Figure 2: Setup of Alfvén wave normal incidence at insulating boundary

where $\delta_w = \sqrt{2\eta_w/\omega} = \sqrt{2/\omega\mu_0\sigma_w}$ is the electromagnetic skin depth of the conductive wall. This gives the general solution in the form of

$$
\begin{aligned}
\mathbf{b}^w &= b_w^+ \hat{\mathbf{y}} \exp\{i(\omega t + k_w z)\} + b_w^- \hat{\mathbf{y}} \exp\{i(\omega t - k_w z)\} \\
&= b_w^+ \hat{\mathbf{y}} \exp\left\{i\left(\omega t + \sqrt{\frac{\omega\mu_0\sigma_w}{2}}(1-i)z\right)\right\} + b_w^- \hat{\mathbf{y}} \exp\left\{i\left(\omega t - \sqrt{\frac{\omega\mu_0\sigma_w}{2}}(1-i)z\right)\right\} \\
&= b_w^+ \hat{\mathbf{y}} \exp\left\{i\left(\omega t + \sqrt{\frac{\omega}{2\eta_w}}(1-i)z\right)\right\} + b_w^- \hat{\mathbf{y}} \exp\left\{i\left(\omega t - \sqrt{\frac{\omega}{2\eta_w}}(1-i)z\right)\right\} \\
&= b_w^+ \hat{\mathbf{y}} \exp\left\{i\left(\omega t + (1-i)\frac{z}{\delta_w}\right)\right\} + b_w^- \hat{\mathbf{y}} \exp\left\{i\left(\omega t - (1-i)\frac{z}{\delta_w}\right)\right\}.
\end{aligned} \tag{43}
$$

Second, now that the continuous magnetic boundary condition cannot be imposed as a homogeneous Dirichlet boundary condition at $z = 0$. Instead, the magnetic boundary condition needs to be split into two conditions, one at $z = 0$, linking $\mathbf{b}$ in the fluid and $\mathbf{b}^w$, the other at $z = h$, linking $\mathbf{b}^w$ to the insulating half space. The two conditions are

$$
b_y|_{z=0^-} = b_y^w|_{z=0^+}, \quad b_y|_{z=h^-} = 0. \tag{44}
$$

The magnetic boundary condition is complemented by the electric boundary condition,

$$
\mathbf{E}|_{z=0^-} = [\eta\nabla\times\mathbf{B} - \mathbf{u}\times\mathbf{B}]|_{z=0^-} = [\eta\nabla\times\mathbf{B}]|_{z=0^+} = \mathbf{E}|_{z=0^+}
$$

which, in the absence of velocity at the boundary, yields

$$
\eta \left.\frac{\partial b_y}{\partial z}\right|_{z=0^-} = \eta_w \left.\frac{\partial b_y}{\partial z}\right|_{z=0^+}. \tag{45}
$$

[Why is continuity of electric field not enforced at the insulating wall? It is not naturally fulfilled, and the electric field at the boundary is actually not zero.] Finally, the kinematic boundary condition, i.e. the no-slip boundary condition, remains the same.

2.2.2 Solutions for magnetic and velocity fields

Substituting the ansatz

$$b_y = \begin{cases} b^+ \exp\left\{i\omega\left(t - \frac{z}{V_A}\right)\right\} + b^- \exp\left\{i\omega\left(t + \frac{z}{V_A}\right)\right\} + b^{\mathrm{BL}} \exp\left\{i\omega t + \frac{V_A z}{\sqrt{\nu\eta}}\right\}, & z < 0 \\ b_w^+ \exp\left\{i\left(\omega t + \sqrt{\frac{\omega}{2\eta_w}}(1-i)z\right)\right\} + b_w^- \exp\left\{i\left(\omega t - \sqrt{\frac{\omega}{2\eta_w}}(1-i)z\right)\right\}, & z > 0 \end{cases} \tag{46}$$

$$u_y = -\frac{b^+}{\sqrt{\rho\mu_0}} \exp\left\{i\omega\left(t - \frac{z}{V_A}\right)\right\} + \frac{b^-}{\sqrt{\rho\mu_0}} \exp\left\{i\omega\left(t + \frac{z}{V_A}\right)\right\} - \frac{b^{\mathrm{BL}}}{\sqrt{\mathrm{Pm}}\sqrt{\rho\mu_0}} \exp\left\{i\omega t + \frac{V_A z}{\sqrt{\nu\eta}}\right\}$$

into the kinematic (no-slip, one scalar eq.) and electromagnetic (two scalar eqs.(44) for the magnetic field and one scalar eq.(45) for the electric field) boundary conditions, we arrive at a system of four linear equations of b^+, b^-, b^{BL}, b_w^+ and b_w^-,

$$\begin{aligned} -\frac{1}{\sqrt{\rho\mu_0}} b^+ + \frac{1}{\sqrt{\rho\mu_0}} b^- - \mathrm{Pm}^{-\frac{1}{2}} \frac{1}{\sqrt{\rho\mu_0}} b^{\mathrm{BL}} &= 0, \\ b^+ + b^- + b^{\mathrm{BL}} &= b_w^+ + b_w^-, \\ b_w^+ \exp\{ik_w h\} + b_w^- \exp\{-ik_w h\} &= 0, \\ -i\frac{\omega}{V_A}\eta b^+ + i\frac{\omega}{V_A}\eta b^- + \sqrt{\frac{\eta}{\nu}} V_A b^{\mathrm{BL}} &= ik_w\eta_w b_w^+ - ik_w\eta_w b_w^-, \end{aligned} \tag{47}$$

which are then rearranged into

$$\begin{aligned} -b^- + \mathrm{Pm}^{-\frac{1}{2}} b^{\mathrm{BL}} &= -b^+ \\ b^- + b^{\mathrm{BL}} - b_w^+ - b_w^- &= -b^+ \\ e^{ik_w h} b_w^+ + e^{-ik_w h} b_w^- &= 0 \\ \frac{\omega}{V_A}\eta b^- - i\frac{V_A}{\sqrt{\mathrm{Pm}}} b^{\mathrm{BL}} - k_w\eta_w b_w^+ + k_w\eta_w b_w^- &= \frac{\omega}{V_A}\eta b^+. \end{aligned} \tag{48}$$

The quantities b^-, b^{BL}, b_w^+ and b_w^- can then be solved as a function of b^+. The most informative quantity here is b^-/b^+, i.e. the reflection coefficient of the boundary. It is also relatively easy to solve. The roadmap to solve this can be summarized as follows. First, from the first equation one can express b^{BL} as a function of b^- and b^+. Using the third equation one can establish the ratio between b_w^+ and b_w^-, which can be used to derive the expression in terms of b^+ and b^-. Finally, we can plug the expressions into the fourth equation, and arrive at

$$R_b = \frac{b^-}{b^+} = \frac{\frac{\omega}{V_A}\eta - iV_A + k_w\eta_w \frac{1+e^{i2k_w h}}{1-e^{i2k_w h}}\left(1 - \sqrt{\mathrm{Pm}}\right)}{\frac{\omega}{V_A}\eta - iV_A - k_w\eta_w \frac{1+e^{i2k_w h}}{1-e^{i2k_w h}}\left(1 + \sqrt{\mathrm{Pm}}\right)} = -\frac{1 - \sqrt{\mathrm{Pm}} - i\frac{V_A}{k_w\eta_w}\left(1 + i\frac{\omega\eta}{V_A^2}\right)\frac{1-e^{i2k_w h}}{1+e^{i2k_w h}}}{1 + \sqrt{\mathrm{Pm}} + i\frac{V_A}{k_w\eta_w}\left(1 + i\frac{\omega\eta}{V_A^2}\right)\frac{1-e^{i2k_w h}}{1+e^{i2k_w h}}}. \tag{49}$$

Recall the travelling wave solution as well as the aforementioned Hartmann layer solution implicitly assumed high Lundquist number, and under this assumption we have

$$S_\omega = \frac{2V_A^2}{\omega(\nu+\eta)} \gg 1, \quad \frac{\omega\eta}{V_A^2} \leq \frac{\omega(\nu+\eta)}{V_A^2} = \frac{2}{S_\omega} \ll 1.$$

Therefore the imaginary term in the bracket scales as ${\mathrm{S}_\omega}^{-1}$, and is negligible at high Lundquist number. The reflection coefficients are then

$$\begin{aligned} R_b &\approx -\frac{1 - \sqrt{\mathrm{Pm}} - i\frac{V_A}{\eta_w k_w}\frac{1-\exp\{i2k_w h\}}{1+\exp\{i2k_w h\}}}{1 + \sqrt{\mathrm{Pm}} + i\frac{V_A}{\eta_w k_w}\frac{1-\exp\{i2k_w h\}}{1+\exp\{i2k_w h\}}} = -\frac{1 - \sqrt{\mathrm{Pm}} - Q(\omega)}{1 + \sqrt{\mathrm{Pm}} + Q(\omega)}, \\ R_u &\approx +\frac{1 - \sqrt{\mathrm{Pm}} - i\frac{V_A}{\eta_w k_w}\frac{1-\exp\{i2k_w h\}}{1+\exp\{i2k_w h\}}}{1 + \sqrt{\mathrm{Pm}} + i\frac{V_A}{\eta_w k_w}\frac{1-\exp\{i2k_w h\}}{1+\exp\{i2k_w h\}}} = +\frac{1 - \sqrt{\mathrm{Pm}} - Q(\omega)}{1 + \sqrt{\mathrm{Pm}} + Q(\omega)}, \end{aligned} \tag{50}$$

where the frequency-dependent complex dimensionless quantity $Q(\omega)$ is given by

$$Q(\omega) = i\frac{V_A}{k_w\eta_w}\frac{1 - e^{i2k_wh}}{1 + e^{i2k_wh}} = \frac{V_A}{k_w\eta_w}\tan k_wh = \frac{V_A\delta_w}{\eta_w}(1-i)\tan\left((1-i)\frac{h}{\delta_w}\right). \tag{51}$$

This is a central dimensionless number in quantifying the effect of finite conductivity at boundary. At the insulating limit, i.e. $\eta_w \to +\infty$ and $k_w \to 0$, we have $Q(\omega) \sim V_A/\eta_w \to 0$, and recover the reflection relation for insulating boundary (eq.42). In general, when $|k_wh| \sim h/\delta_w \ll 1$, the linear approximation of the exponential can be used, which yields

$$Q(\omega) \approx i\frac{V_A}{k_w\eta_w}\frac{-i2k_wh}{2} = \frac{V_Ah}{\eta_w} = \mu_0V_A\sigma_wh = \sqrt{\frac{\mu_0}{\rho}}B_0G \tag{52}$$

where $G = \sigma_w\delta$ is the total conductance in the layer. This approximation is coined the thin-layer approximation, since it assumes the wall thickness δ to be much smaller than the skin depth in the wall δ_w. At this limit the quantity Q is frequency-independent.

For completeness, other quantities of interest are related to the reflection coefficients via

$$\begin{aligned}
\frac{b^{\mathrm{BL}}}{b^+} &= \sqrt{\mathrm{Pm}}\,(R_b - 1) = -\frac{2\sqrt{\mathrm{Pm}}}{1+\sqrt{\mathrm{Pm}}+Q(\omega)}, \\
\frac{b_w^+}{b^+} &= \frac{1}{1-e^{i2k_wh}}\left(1-\sqrt{\mathrm{Pm}} + (1+\sqrt{\mathrm{Pm}})R_b\right) = \frac{1}{1-e^{i2k_wh}}\frac{2Q(\omega)}{1+\sqrt{\mathrm{Pm}}+Q(\omega)}, \\
\frac{b_w^-}{b^+} &= \frac{-e^{i2k_wh}}{1-e^{i2k_wh}}\left(1-\sqrt{\mathrm{Pm}} + (1+\sqrt{\mathrm{Pm}})R_b\right) = \frac{-e^{i2k_wh}}{1-e^{i2k_wh}}\frac{2Q(\omega)}{1+\sqrt{\mathrm{Pm}}+Q(\omega)}.
\end{aligned} \tag{53}$$

Electromagnetic boundary condition at insulating boundary

Here I try to start from first principles and derive the electromagnetic boundary condition at an insulating wall. The fact that only the continuity of magnetic field is enforced at the interface is not well justified at first glance, and would require more than intuition to understand.

Regardless of the specific setting, for the interface where a conductor comes in contact with an insulator, the current density in the system is always finite; in other words, there is no electric currents that can be concentrated in an infinitely thin layer, i.e. no current sheet. This leads to the following continuities at the boundary

$$\begin{aligned}
\hat{\mathbf{n}}\cdot\mathbf{B}|_{\Sigma^-} &= \hat{\mathbf{n}}\cdot\mathbf{B}|_{\Sigma^+} \\
\hat{\mathbf{n}}\times\mathbf{B}|_{\Sigma^-} &= \hat{\mathbf{n}}\times\mathbf{B}|_{\Sigma^+} \\
\hat{\mathbf{n}}\times\mathbf{E}|_{\Sigma^-} &= \hat{\mathbf{n}}\times\mathbf{E}|_{\Sigma^+}.
\end{aligned} \tag{54}$$

These indicate continuity of magnetic field, and the continuity of tangent electric field. In general, there might be discontinuity in the normal electric field at conductor-insulator boundaries, which would then require sheet electric charges. In the 1-D scenarios of Alfvén waves presented here, however, there is no electric field normal to the boundary.

First, let us consider the interface between an electrically conductive solid and an insulating solid. In the conductive region, we assume that the temporal variation is relatively slow so that the displacement current can be neglected. The electromagnetic fields are then related via

$$\nabla\times\mathbf{E} = -i\omega\mathbf{B}, \qquad \nabla\times\mathbf{B} = \sigma\mu_0\mathbf{E} = \frac{1}{\eta}\mathbf{E}.$$

In the insulating region, the electromagnetic fields are related via

$$\nabla\times\mathbf{E} = -i\omega\mathbf{B}, \qquad \nabla\times\mathbf{B} = i\omega\varepsilon_0\mu_0\mathbf{E} = i\frac{\omega}{c^2}\mathbf{E}.$$

The two different mechanisms result in very different characteristic length scales of variation. In conductive medium where the diffusive term dominates, the length scale is given by the skin depth

$$L_{\mathrm{cond}} = \delta_s = \sqrt{\frac{2\eta}{\omega}} = \sqrt{\frac{2}{\omega\mu_0\sigma}}$$

while in insulating medium where the wave term dominates, the length scale is the wavelength

$$L_{\text{insl}} = \bar{\lambda} = \frac{c}{\omega} = \frac{1}{\omega\sqrt{\mu_0\varepsilon_0}}.$$

The distinctive length scale results in fundamentally different amplitude distributions among electric and magnetic field.

$$\begin{aligned} \left(\frac{B}{E}\right)_{\text{cond}} &\sim \frac{1}{\omega L_{\text{cond}}} \sim \frac{L_{\text{cond}}}{\eta} \sim \frac{1}{\sqrt{\omega\eta}} \\ \left(\frac{B}{E}\right)_{\text{insl}} &\sim \frac{1}{\omega L_{\text{insl}}} \sim \frac{\omega L_{\text{insl}}}{c^2} \sim \frac{1}{c}. \end{aligned} \tag{55}$$

Invoking the dimensionless quantity

$$\epsilon = \frac{|\varepsilon_0 \partial_t \mathbf{E}|}{|\sigma \mathbf{E}|} \sim \frac{\omega\eta}{c^2} = \frac{\omega\varepsilon_0}{\sigma}$$

which is again the quantity that thresholds whether displacement current can be neglected or not, we see that if the electric fields are of matching amplitudes on both sides, then the magnetic fields are of relative magnitudes

$$\left(\frac{B_{\Sigma^+}}{B_{\Sigma^-}}\right) \approx \sqrt{\epsilon} \tag{56}$$

The light speed $c \approx 3\times 10^8 \mathrm{m}\cdot\mathrm{s}^{-1}$. For scenarios relevant to the Earth core, $\eta \approx 1\mathrm{m}^2\cdot\mathrm{s}^{-1}$. Therefore, even processes that vary on the scale of seconds has values $\epsilon \sim 10^{-17}$; processes that are of lower frequencies have even lower ϵ. In short, the magnetic field in the insulating wall derived this way will be very much marginal compared to the field in the conductive counterpart. Viewed from the conductive medium, this is practically zero.

[This apparently holds in plane waves, but why does this relation not hold in spherical geometry? In geodynamo simulations, it seems the boundary condition for poloidal field is not homogeneous Dirichlet BC.]

To illustrate the effect, I redo the problem of Alfvén waves impinging on an insulating wall, this time with electric boundary condition included. At the same time, the solution in the insulating wall should also be constructed, which takes the form of

$$b_y = b_w \exp\left\{i\omega\left(t - \frac{z}{c}\right)\right\}.$$

The three boundary conditions thus yield

$$\begin{aligned} b^+ + b^- + b^{\text{BL}} &= b_w, \\ b^+ - b^- + \text{Pm}^{-\frac{1}{2}} b^{\text{BL}} &= 0, \\ -i\eta\frac{\omega}{V_A}b^+ + i\eta\frac{\omega}{V_A}b^- + \sqrt{\frac{\eta}{\nu}}V_A b^{\text{BL}} &= -cb_w \end{aligned}$$

The reflection coefficient in this case is given by

$$R_b = \frac{b^-}{b^+} = -\frac{1 - \sqrt{\text{Pm}} - \left(i\frac{\omega\eta}{cV_A} + \frac{V_A}{c}\right)}{1 + \sqrt{\text{Pm}} + \left(i\frac{\omega\eta}{cV_A} + \frac{V_A}{c}\right)} = -\frac{1 - \sqrt{\text{Pm}} - \beta_A\left(1 + iS_\eta^{-1}\right)}{1 + \sqrt{\text{Pm}} + \beta_A\left(1 + iS_\eta^{-1}\right)}, \tag{57}$$

where S_η is the frequency-dependent Lundquist number for magnetic diffusion, and $\beta_A = V_A/c$ is the ratio between Alfvén speed and light speed. Since Alfvén waves and electromagnetic waves operate at fundamentally different speeds ($\beta_A \sim 10^{-11} - 10^{10}$), the modification to the original formulae is negligible.

3 Boundary layers oscillating along the boundary

We have seen in the derivation of Hartmann layers that viscous and magnetic diffusion results in a boundary layer solution decaying exponentially in space, whose decay rate is dependent on the angle between **k** and $\mathbf{B}_0$. This expression comes in handy when the "direction" is well-defined, i.e. **k** can be written as some quantity k (whether real or complex) times some real unit vector. However, this means the derived relation is most interpretable when the decaying and the oscillatory behaviour is uniform in all directions. In other words, the spatial part of the exponent would take the form (I take Lundquist number $S_\omega \to +\infty$ and the wave vector in Oxz plane just to simplify the expression)

$$\exp\left\{\mp\frac{V_A\cos\gamma}{\sqrt{\nu\eta}}\left(\sin\theta x+\cos\theta z\right)\right\}=\exp\left\{\mp\left(\sin\theta\frac{x}{\delta_{\mathrm{BL}}}+\cos\theta\frac{z}{\delta_{\mathrm{BL}}}\right)\right\}$$

which shows if the wave decays in z-direction (takes the minus sign and take $\theta\in(0,\pi/2)$), it also decays in the positive x-direction, and grows exponentially in the negative x-direction. Similarly, when analyizing the electromagnetic wave in a conductive medium (see the first box in the next section), we have the skin depth δ_s, which gives the spatial dependency

$$\exp\left\{\mp(i+1)\sqrt{\frac{\omega}{2\eta}}(\sin\theta x+\cos\theta z)\right\}=\exp\left\{\mp(i+1)\left(\sin\theta\frac{x}{\delta_s}+\cos\theta\frac{z}{\delta_s}\right)\right\}.$$

Unless $\theta=0$ as in normal incidence (the case for Schaeffer, Jault, et al. 2012 and Schaeffer and Jault 2016), neither of these solutions would be valid for our plane wave analysis at some real θ. The problem is that in plane wave analysis, it is forbidden to have a solution that grows to infinity at $x=\infty$ (infinity at the boundary). **Therefore we have to seek solution that is bounded at $x=\infty$.** A simple way out of this is to state $\cos\theta\in\mathbb{C}$. This works well for the electromagnetic wave in a conductive medium, but not so much in the Alfvén wave case, for the latter contains an additional variable γ which is dependent on θ.

In this section I shall formulate the solutions by postulating **a fixed horizontal slowness**, denoted as $p=\omega/k_x$. This is a preserved quantity in any analysis of reflection and refraction of plane waves. For the phases of different wave components to match at the interface, the equivalent condition is that all wave components share the same horizontal (or interface-parallel) wavenumber. The plane waves now have spatial dependency in the form

$$\exp\left\{i(\omega px+k_z z)\right\}.$$

The treatment as well as the dispersion relations developed in this section will be used for treating oblique incidence cases.

3.1 Electromagnetic wave in conductive medium

We state the Helmholtz equation for electromagnetic wave in conductive medium

$$\nabla^2\mathbf{B}-\frac{i\omega}{\eta}\mathbf{B}=\nabla^2\mathbf{B}-i\omega\mu_0\sigma\mathbf{B}=0.$$

Postulating the solution of the form

$$\mathbf{B}=\mathbf{B}_1\exp\left\{i\omega\left(t-px-\frac{k_z}{\omega}z\right)\right\}\qquad(p\in\mathbb{R}),$$

we arrive at the dispersion relation for k_z

$$k_z^2=-\omega^2p^2-i\frac{\omega}{\eta}=-\omega^2p^2-i\omega\mu_0\sigma. \tag{58}$$

The solution from the dispersion relation is

$$k_z=\pm\left(\frac{-\omega^2p^2+\sqrt{\omega^4p^4+\omega^2/\eta^2}}{2}\right)^{\frac{1}{2}}\mp i\left(\frac{\omega^2p^2+\sqrt{\omega^4p^4+\omega^2/\eta^2}}{2}\right)^{\frac{1}{2}}. \tag{59}$$

It is not immediately clear which term is more significant. In the context of Alfvén waves, the horizontal slowness may be given by $p = \sin\theta / V_A \cos\gamma \sim 1/V_A$ (when $|\theta - n\pi| > \epsilon$ and $|\gamma - (n+1/2)\pi| > \epsilon$). We can take the ratio between ω/η and $\omega^2 p^2$, and write

$$\frac{\omega/\eta}{\omega^2 p^2} = \frac{2/\delta_s^2}{k_x^2} = 2\frac{\bar{\lambda}_x^2}{\delta_s^2}, \qquad \frac{\omega/\eta}{\omega^2 p^2} = \frac{1}{\omega\eta p^2} = \frac{V_A^2}{\omega\eta} = \frac{\sigma B_0^2}{\omega\rho}$$

which is the "Lundquist" number with only magnetic diffusion, and the magnetic diffusivity in the conductive wall is used instead of diffusivity in the fluid. If we take $\omega \sim 2\pi/(6\times\pi\times 10^7)\mathrm{s}^{-1} \approx 3\times 10^{-8}\mathrm{s}^{-1}$ (on the ground of the LOD variation, etc.), $\rho \sim 10^4 \mathrm{kg/m}^3$ (rough value well constrained based on gravity, seismology, mineralogy, etc.), and magnetic field $\sim 10^{-3}\mathrm{T}$ (arguable?), then this quantity is

$$\frac{\omega/\eta}{\omega^2 p^2} \sim \frac{\sigma}{300\mathrm{S/m}}.$$

While for the core where $\sigma \sim 2\times 10^5\mathrm{S}$, the Lundquist number is justifiably large, at the order of 10^3, for the lower mantle, if one uses $\sigma \approx 10^1 - 10^3\mathrm{S}$, the quantity is

$$\frac{\omega/\eta}{\omega^2 p^2} \approx 0.03 - 3$$

which is within 0 to 1 order of magnitude around unity. Therefore, it is really unclear which one is the smaller one, and how to expand the square roots into Taylor series. That being said, I shall still try to simplify the equations in two endmember cases. First, for highly conductive medium or small horizontal wavenumber (large horizontal wavelength) with $\omega/\eta \gg \omega^2 p^2$, we have

$$k_z \approx \pm\sqrt{\frac{\omega}{2\eta}}\left[\left(1 - \frac{1}{2}\omega\eta p^2\right) - i\left(1 + \frac{1}{2}\omega\eta p^2\right)\right], \tag{60}$$

which reduces to the simple form of $k_w = \pm(1-i)/\delta_s$ at normal incidence or $p = 0$. As the horizontal wavenumber in the x direction grows from 0 to a small value, the decay rate mildly grows in the z direction, while the spatial oscillation rate mildly drops. For the other endmember, highly resistive medium or large horizontal wavenumber (small horizontal wavelength), the z wavenumber is simplified into

$$k_z \approx \pm\omega p\left[\frac{1}{2\omega^2\eta^2 p^4} - i\left(1 + \frac{1}{8\omega^2\eta^2 p^4}\right)\right]. \tag{61}$$

The wave solution then decays in the z-direction at the same length scale as the horizontal wavelength, while the oscillation in the z-direction has much larger wavelength. As conductivity decreases, the wavelength of oscillation increases, until in the resistive limit this goes to infinity, degenerating into the evanescent solution of the Laplace equation.

3.2 Electromagnetic wave in insulator

We state the wave equation for electromagnetic wave in the insulating medium

$$\nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = \mathbf{0} \tag{62}$$

which is obtained by keeping the displacement current in the Ampere's law. If the temporal variation is very small, in other words, if the wavelength of the electromagnetic wave $\sim c/\omega$ is much larger than the characteristic length scale of magnetic field variation, then the system is "quasi-static", and the field fulfills the Laplace equation $\nabla^2\mathbf{B} = 0$. Equivalently, the magnetic field can be expressed through a scalar potential $\mathbf{B} = -\nabla V$, and the scalar potential fulfills the scalar Laplace equation. The implicit assumption would be that the propagation of waves is so fast, that the variation of the field is propagated almost instantaneously (compared to the variation time at the boundary) across the characteristic length scales.

Therefore, despite its simplicity, the Laplace equation cannot resolve the propagation of EM waves, and will be problematic when one analyzes the energy flux in the system. Using the same wave ansatz, we have from the wave equation

$$k_z^2 + k_x^2 = \frac{\omega^2}{c^2} \quad \Longrightarrow \quad k_z = \pm\omega\sqrt{\frac{1}{c^2} - p^2}. \tag{63}$$

For very small $p < 1/c$, we still have $k_z \in \mathbb{R}$. However, as soon as $p > 1/c$, the wave becomes evanescent in the z direction. To develop a feeling for this threshold, we take $c = 3 \times 10^8 \mathrm{m} \cdot \mathrm{s}^{-1}$. If the frequency is approximately 1Hz, the critical horizontal wavelength is $\lambda_c \approx 3 \times 10^8$m. Any horizontal wavelength smaller than this would give rise to an evanescent electromagnetic wave. Threshold on the horizontal wavelength for lower frequencies would be even larger. In terms of incidence angle of the Alfvén waves, if we take $V_A \sim 0.1 \mathrm{m} \cdot \mathrm{s}^{-1}$, any $\theta >\sim 10^{-9}$rad would give rise to evanescent waves. It is fairly reasonable to say that, in the parameter space that remotely resembles the Earth's core, only Alfvén waves that are almost normal incidence can excite travelling electromagnetic waves

$$\mathbf{B} \exp\left\{i\omega\left(t - px - \sqrt{\frac{1}{c^2} - p^2}z\right)\right\}, \tag{64}$$

otherwise the matching electromagnetic wave is just an evanescent wave

$$\mathbf{B} \exp\left\{i\omega(t - px) - \omega\sqrt{p^2 - \frac{1}{c^2}}z\right\}. \tag{65}$$

When $p \gg 1/c$, the length scale over which the wave decays is the same as the horizontal wavelength.

3.3 Alfvén wave and Hartmann layer solution

We recall the coupled equation for Alfvén wave in diffusive medium

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} &= \frac{\mathbf{B}_0 \cdot \nabla}{\rho\mu_0}\mathbf{b} + \nu\nabla^2\mathbf{u}, \\
\frac{\partial \mathbf{b}}{\partial t} &= \mathbf{B}_0 \cdot \nabla\mathbf{u} + \eta\nabla^2\mathbf{b}.
\end{aligned}$$

Postulating the solution of the form

$$\mathbf{u} = \mathbf{u}_0 \exp\left\{i\omega\left(t - px - \frac{k_z}{\omega}z\right)\right\}, \quad \mathbf{b} = \mathbf{b}_0 \exp\left\{i\omega\left(t - px - \frac{k_z}{\omega}z\right)\right\} \qquad (p \in \mathbb{R}),$$

we arrive at the equations in frequency-wavenumber domain

$$\begin{aligned}
(i\omega + \nu(k_z^2 + \omega^2 p^2))\mathbf{u} + i\frac{B_x\omega p + B_z k_z}{\rho\mu_0}\mathbf{b} &= \mathbf{0} \\
i(B_x\omega p + B_z k_z)\mathbf{u} + (i\omega + \eta(k_z^2 + \omega^2 p^2))\mathbf{b} &= \mathbf{0}
\end{aligned}$$

which yields the dispersion relation

$$-\omega^2 + i\omega(\nu + \eta)\left(\omega^2 p^2 + k_z^2\right) + \nu\eta\left(\omega^2 p^2 + k_z^2\right)^2 + \frac{1}{\rho\mu_0}(B_x\omega p + B_z k_z)^2 = 0. \tag{66}$$

We see the resemblance with eq.(13), except here $\mathbf{k}$ is replaced with $\omega p\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}$, and so k^2 is replaced with $\omega^2 p^2 + k_z^2$. We also see once again that the sole element that breaks the isotropy is the background field $\mathbf{B}_0$. However, different from eq.(13) which yields the biquadratic form of k, this dispersion relation is fully quartic, or of degree four, in k_z:

$$\begin{aligned} \nu\eta k_z^4 + \left(2\nu\eta\omega^2 p^2 + i\omega(\nu+\eta) + \frac{B_z^2}{\rho\mu_0}\right) k_z^2 + \frac{2B_x B_z}{\rho\mu_0}\omega p k_z \\ + \left(-\omega^2 + \nu\eta\omega^4 p^4 + i(\nu+\eta)\omega^3 p^2 + \frac{B_x^2}{\rho\mu_0}\omega^2 p^2\right) = 0 \\ \left(\frac{k_z}{\omega p}\right)^4 + \left(2 + i\frac{\nu+\eta}{\nu}\frac{1}{\omega\eta p^2} + \frac{B_z^2}{\rho\mu_0\omega\eta}\frac{1}{\omega\nu p^2}\right)\left(\frac{k_z}{\omega p}\right)^2 + 2\frac{B_x B_z}{\rho\mu_0\omega\eta}\frac{1}{\omega\nu p^2}\left(\frac{k_z}{\omega p}\right) \\ + \left(-\frac{1}{\omega^2 p^4 \nu\eta} + 1 + i\frac{\nu+\eta}{\nu}\frac{1}{\omega\eta p^2} + \frac{B_x^2}{\rho\mu_0\omega\eta}\frac{1}{\omega\nu p^2}\right) = 0 \end{aligned} \tag{67}$$

We see that somehow $1/\omega\eta p^2$ is popping up frequently in the equation. This quantity is no other than $2\bar{\lambda}_x^2/\delta_s^2$, or $\frac{V_A^2}{\omega\eta}\frac{\cos\gamma}{\sin\theta}$, as previously shown. Therefore this quantity indicates the relative length scale of x-wavelength compared to the skin depth inside the fluid medium, and can be defined as another Lundquist number, denoted by S_p. Nondimensionalizing the equation with the quantities,

$$\tilde{k}_z = \frac{k_z}{\omega p}, \quad S_\eta = \frac{V_A^2}{\omega\eta}, \quad S_p = \frac{1}{\omega\eta p^2}, \quad \mathrm{Pm} = \frac{\nu}{\eta}$$

we come to the dimensionless form of the equation

$$\begin{aligned} \tilde{k}_z^4 + \left(2 + i\frac{1+\mathrm{Pm}}{\mathrm{Pm}}S_p + \frac{S_\eta S_p}{\mathrm{Pm}}\sin^2\alpha\right)\tilde{k}_z^2 + \frac{S_\eta S_p}{\mathrm{Pm}}\sin 2\alpha \tilde{k}_z \\ + \left(1 - \frac{S_p^2}{\mathrm{Pm}} + i\frac{1+\mathrm{Pm}}{\mathrm{Pm}}S_p + \frac{S_\eta S_p}{\mathrm{Pm}}\cos^2\alpha\right) = 0 \end{aligned} \tag{68}$$

where α is the azimuthal angle of $\mathbf{B}_0$ within the Oxy plane.

Alternatively, the z-wavenumber can also be nondimensionalized with the characteristic wavelength of Alfvén wave, i.e.

$$\tilde{k}_z = \frac{k_z}{k_A} = k_z\bar{\lambda}_A = \frac{k_z V_A}{\omega} = k_z\frac{B_0}{\omega\sqrt{\rho\mu_0}}.$$

Down this route, the dispersion relation is nondimensionalized as follows

$$\begin{aligned} \left(\frac{k_z}{k_A}\right)^4 + \left(2V_A^2 p^2 + i\frac{\nu+\eta}{\nu}\frac{V_A^2}{\omega\eta} + \frac{V_A^4}{\omega^2\nu\eta}\sin^2\alpha\right)\left(\frac{k_z}{k_A}\right)^2 + \frac{V_A^4}{\omega^2\nu\eta}pV_A\sin 2\alpha\left(\frac{k_z}{k_A}\right) \\ + \left(V_A^4 p^4 + i\frac{\nu+\eta}{\nu}\frac{V_A^2}{\omega\eta}V_A^2 p^2 + \frac{V_A^4}{\omega^2\nu\eta}V_A^2 p^2\cos^2\alpha - \frac{V_A^4}{\omega^2\nu\eta}\right) = 0 \end{aligned} \tag{69}$$

Expressing again the dimensionless groups in terms of Lundquist number for magnetic diffusion ($S_\eta = V_A^2/\omega\eta$), magnetic Prandtl number ($\mathrm{Pm} = \nu/\eta$), and $\bar{p} = pV_A$ (for travelling waves this is simply the ratio between two angle cosines, which is at order 1 for most configurations, see the box below), we arrive at the nondimensional equation

$$\begin{aligned} \tilde{k}_z^4 + \left(2\bar{p}^2 + i\frac{1+\mathrm{Pm}}{\mathrm{Pm}}S_\eta + \frac{S_\eta^2}{\mathrm{Pm}}\sin^2\alpha\right)\tilde{k}_z^2 + \frac{S_\eta^2}{\mathrm{Pm}}\bar{p}\sin 2\alpha\tilde{k}_z \\ + \left(\bar{p}^4 + i\frac{1+\mathrm{Pm}}{\mathrm{Pm}}S_\eta\bar{p}^2 + \frac{S_\eta^2}{\mathrm{Pm}}\bar{p}^2\cos^2\alpha - \frac{S_\eta^2}{\mathrm{Pm}}\right) = 0 \end{aligned} \tag{70}$$

The horizontal slowness in travelling Alfvén wave

Here I present a very concise discussion on the magnitude of horizontal slowness in travelling Alfvén waves. The derivations in the following sections will be more elaborate, but not catered to this parameterization. I consider the scenario where background field is coplanar with the plane of incidence (the out-of-plane component actually does not matter), both in the Oxz-plane, and the Alfvén wave takes the form

$$\mathbf{b} = \mathbf{b}_0 \exp\left\{i\omega\left(t - px - qz\right)\right\}$$

Since the Alfvén wave is travelling, I shall take $p, q \in \mathbb{R}$. The wave propagates in the $(\sin\theta, 0, \cos\theta)$ direction, and the background field is in the direction $(\cos\alpha, 0, \sin\alpha)$. The dispersion relation gives (approximately, ignoring the finite Lundquist number term)

$$\omega p \frac{B_x}{|\mathbf{B}_0|} + \omega q \frac{B_z}{|\mathbf{B}_0|} = \pm\frac{\omega}{V_A} \quad \Longrightarrow \quad pV_A \cos\alpha + qV_A \sin\alpha = \pm 1$$

Since the angle θ gives the angle between the wave vector and the z axis, it can be seen that $q/p = k_z/k_x = \cot\theta$. Plugging it into the expression, we have the expression for pV_A

$$pV_A = \pm\frac{1}{\cot\theta\sin\alpha + \cos\alpha} = \pm\frac{\sin\theta}{\cos\theta\sin\alpha + \sin\theta\cos\alpha} = \pm\frac{\hat{\mathbf{k}}\cdot\hat{\mathbf{x}}}{\hat{\mathbf{k}}\cdot\hat{\mathbf{B}}_0}. \tag{71}$$

For most configurations with wave vector not too close to $\hat{\mathbf{z}}$ and not too close to normal of $\hat{\mathbf{B}}_0$ (in which case there is scarcely any propagation anyway (group velocity very small)), the quantity is close to unity. It can be also seen in fig.(3), that the scenario where $\bar{p}$ is one order of magnitude away from unity (especially $\bar{p} > 10$) occupies a very small domain in the parameter space.

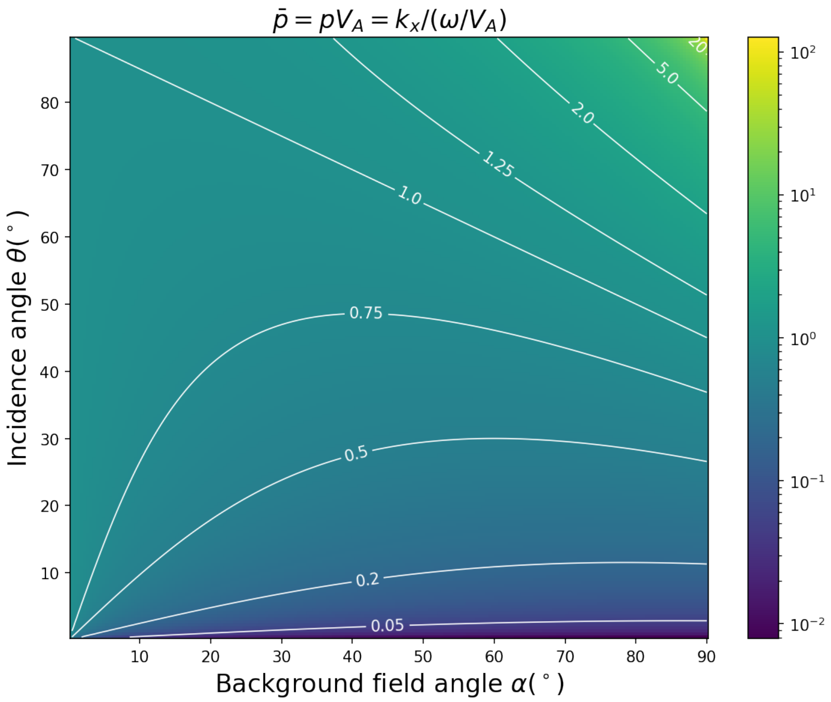

Figure 3: Dimensionless horizontal slowness $\bar{p}$ as a function of $\mathbf{B}_0 - \hat{\mathbf{x}}$ angle α and incidence angle θ.

I shall use expression eq.(70) in favour of eq.(68), since eq.(68) introduces artificially a singularity at $p \to 0$, i.e. normal incidence. At this limit it is no longer legimate to normalize the z-wavenumber with the x-wavenumber since the latter is trivial. On the other hand, eq.(70) degenerates to the dispersion relation eq.(13), at normal incidence $\bar{p} = 0$ (note in this case $\sin\alpha = \cos\gamma$).

Although general solution to any quartic equation exists, and so the roots of the dispersion relation can, in principle, be expressed analytically, the formulae would be somewhat lengthy and offers virtually no additional insights. I shall investigate the solution in two ways. First, I shall take a pertubative approach and look at the asymptotic behaviour of the solution at high Lundquist number, expressing the solution as a series in inverse Lundquist number. Second, I shall calculate numerically the solution to the dispersion relation, and see how the solved k_z compares with the first-order expansion.

3.4 Spatial branch of MHD waves at high Lundquist number

3.4.1 Perturbative solution to first order

While solving analytically eq.(70) yields very lengthy solution, prohibiting meaningful interpretation of the result, it is however feasible to characterize the spatial branch solution at high Lundquist number. More specifically, we wish to expand the solution to leading orders of inverse Lundquist number. This motivates a pertubative approach to solving the equation. We know *a priori* that the solution to eq.(70) should comprise of both travelling Alfvén waves and Hartmann boundary layers. To begin the derivation, let us first examine the simplified systems that renders these endmember solutions. The travelling Alfvén wave dispersion relation can be obtained once we take the ideal limit, where $S_\eta \to +\infty$. With finite $\tilde{k}_z$, the equation can be simplified by taking all the terms that contain S_η^2:

$$\begin{aligned} &\frac{S_\eta^2}{\mathrm{Pm}}\left[\sin^2\alpha\tilde{k}_z^2 + 2\bar{p}\sin\alpha\cos\alpha\tilde{k}_z + \bar{p}^2\cos^2\alpha - 1\right] = 0 \\ &\left(\sin\alpha\tilde{k}_z + \bar{p}\cos\alpha\right)^2 = 1 \quad \Longrightarrow \quad \tilde{k}_{1,2} = \frac{\pm 1 - \bar{p}\cos\alpha}{\sin\alpha} \end{aligned} \tag{72}$$

which yields the diffusionless travelling Alfvén wave solution. This can be perceived as a zeroth order solution in S_η^{-1}. On the other hand, the Hartmann boundary layer cannot be obtained in this manner, as this solution scales with S_η^2, and goes to infinity together with S_η at high Lundquist number limit. We must then take the scaling $\tilde{k}_z \sim S_\eta^2$, and keep the leading order term in S_η in eq.(70), which gives

$$\tilde{k}_z^4 + \frac{S_\eta^2}{\mathrm{Pm}}\sin^2\alpha\tilde{k}_z = \tilde{k}_z^2\left(\tilde{k}_z^2 + \frac{S_\eta^2}{\mathrm{Pm}}\sin^2\alpha\right) = 0 \quad \Longrightarrow \quad \tilde{k}_{3,4} = \pm i\frac{S_\eta}{\sqrt{\mathrm{Pm}}}\sin\alpha \tag{73}$$

the Hartmann layer solution. The trivial solutions are already neglected (formally these correspond to the travelling Alfvén waves, but they are negligible in $\tilde{k}_z$ magnitude compared to the Hartmann layer, hence trivial). This should also be perceived as a zeroth order solution. The two simplification approaches lead to two unperturbed equations, whose solutions can be easily derived and correspond to Alfvén wave and Hartmann layer, respectively. The question now is, how will the additional terms that are neglected in these simplifications modify the solution $\tilde{k}_z$? To this end, a pertubative approach can be employed. The machinery to be used is documented in Appendix C.

To employ this machinery, we consider the original equation split into an unperturbed form and perturbation terms. For the travelling Alfvén wave, the unperturbed and perturbation polynomials are given by

$$\begin{aligned} p_0(\tilde{k}_z) &= \frac{S_\eta^2}{\mathrm{Pm}}\left[\left(\tilde{k}_z\sin\alpha + \bar{p}\cos\alpha\right)^2 - 1\right] \\ \delta p(\tilde{k}_z) &= \tilde{k}_z^4 + \left(iS_\eta\frac{1+\mathrm{Pm}}{\mathrm{Pm}} + 2\bar{p}^2\right)\tilde{k}_z^2 + \left(iS_\eta\frac{1+\mathrm{Pm}}{\mathrm{Pm}} + \bar{p}^2\right)\bar{p}^2 \end{aligned} \tag{74}$$

and the unperturbed solutions are given by eq.(72). According to Appendix C, the leading order correction to the roots are given by

$$
\begin{aligned}
\delta\tilde{k}_{1,2} &= -\frac{\delta p(\tilde{k}_{1,2})}{p_0'(\tilde{k}_{1,2})} \\
&= \mp\frac{\mathrm{Pm}}{2S_\eta^2\sin\alpha}\left[\left(\frac{\pm 1-\bar{p}\cos\alpha}{\sin\alpha}\right)^4+\left(iS_\eta\frac{1+\mathrm{Pm}}{\mathrm{Pm}}+2\bar{p}^2\right)\left(\frac{\pm 1-\bar{p}\cos\alpha}{\sin\alpha}\right)^2+\left(iS_\eta\frac{1+\mathrm{Pm}}{\mathrm{Pm}}+\bar{p}^2\right)\bar{p}^2\right] \\
&\approx \mp i\frac{1+\mathrm{Pm}}{2S_\eta\sin\alpha}\left[\left(\frac{\pm 1-\bar{p}\cos\alpha}{\sin\alpha}\right)^2+\bar{p}^2\right]=\mp i\frac{1+\mathrm{Pm}}{2S_\eta\sin\alpha}\frac{1\mp 2\bar{p}\cos\alpha+\bar{p}^2}{\sin^2\alpha}.
\end{aligned}
\tag{75}
$$

Note in the last line, only terms in the leading order of S_η are kept. It follows that the travelling Alfvén wave with first order correction is

$$
\begin{aligned}
\tilde{k}_{1,2}' &= \frac{\pm 1-\bar{p}\cos\alpha}{\sin\alpha}\mp i\frac{1+\mathrm{Pm}}{2S_\eta\sin^3\alpha}\left(1\mp 2\bar{p}\cos\alpha+\bar{p}^2\right)+O\left(S_\eta^{-2}\right) \\
&= \frac{\pm 1-\bar{p}\cos\alpha}{\sin\alpha}\left[1-i\frac{1+\mathrm{Pm}}{2S_\eta\sin^2\alpha}\frac{1\mp 2\bar{p}\cos\alpha+\bar{p}^2}{1\mp\bar{p}\cos\alpha}+O\left(S_\eta^{-2}\right)\right].
\end{aligned}
\tag{76}
$$

Similarly, the original equation can be split into an unperturbed form that yields the Hartmann layer solution, and the perturbation terms:

$$
\begin{aligned}
p_0(\tilde{k}_z) &= \tilde{k}_z^4+\frac{S_\eta^2}{\mathrm{Pm}}\sin^2\alpha\tilde{k}_z^2, \\
\delta p(\tilde{k}_z) &= \left(iS_\eta\frac{1+\mathrm{Pm}}{\mathrm{Pm}}+2\bar{p}^2\right)\tilde{k}_z^2+\frac{S_\eta^2}{\mathrm{Pm}}\bar{p}\sin 2\alpha\tilde{k}_z+\left(\frac{S_\eta^2}{\mathrm{Pm}}\left(\bar{p}^2\cos^2\alpha-1\right)+\left(iS_\eta\frac{1+\mathrm{Pm}}{\mathrm{Pm}}+\bar{p}^2\right)\bar{p}^2\right)
\end{aligned}
\tag{77}
$$

and the unperturbed solutions are given by eq.(73). The leading order correction to the roots are

$$
\begin{aligned}
\delta\tilde{k}_{3,4} &= -\frac{\delta p(\tilde{k}_{3,4})}{p_0'(\tilde{k}_{3,4})}=-\frac{1}{2}\left(\pm i\frac{S_\eta}{\sqrt{\mathrm{Pm}}}\sin\alpha\right)^{-3}\delta p(\tilde{k}_{3,4}) \\
&\approx \pm\frac{\mathrm{Pm}^{\frac{3}{2}}}{i2S_\eta^3\sin^3\alpha}\left[-iS_\eta\frac{1+\mathrm{Pm}}{\mathrm{Pm}}\frac{S_\eta^2}{\mathrm{Pm}}\sin^2\alpha\pm i\frac{S_\eta^3}{\mathrm{Pm}^{\frac{3}{2}}}2\bar{p}\sin^2\alpha\cos\alpha\right] \\
&= \pm\frac{1}{2\sin\alpha}\left[-\frac{1+\mathrm{Pm}}{\sqrt{\mathrm{Pm}}}\pm 2\bar{p}\cos\alpha\right].
\end{aligned}
\tag{78}
$$

Similarly, only the terms in the leading order of S_η are kept. The corrected Hartmann layer solution is

$$
\begin{aligned}
\tilde{k}_{3,4}' &= \pm i\frac{S_\eta}{\sqrt{\mathrm{Pm}}}\sin\alpha\pm\frac{1}{2\sin\alpha}\left(-\frac{1+\mathrm{Pm}}{\sqrt{\mathrm{Pm}}}\pm 2\bar{p}\cos\alpha\right)+O\left(S_\eta^{-1}\right) \\
&= \pm i\frac{S_\eta}{\sqrt{\mathrm{Pm}}}\sin\alpha\left[1+i\frac{1+\mathrm{Pm}\mp 2\bar{p}\sqrt{\mathrm{Pm}}\cos\alpha}{2S_\eta\sin^2\alpha}+O\left(S_\eta^{-2}\right)\right].
\end{aligned}
\tag{79}
$$

As a quick verification, we can plut in $\bar{p}=0$, corresponding to normal incidence. Restoring the wavenumbers to their dimensional form, we have

$$
\begin{aligned}
k_{1,2} &= \pm\frac{\omega}{V_A\sin\alpha}\left[1-i\frac{1+\mathrm{Pm}}{2S_\eta\sin^2\alpha}+O\left(S_\eta^{-2}\right)\right]=\pm\frac{\omega}{V_A\sin\alpha}\left[1-i\frac{\omega(\nu+\eta)}{2V_A^2\sin^2\alpha}+O\left(S_\eta^{-2}\right)\right] \\
k_{3,4} &= \pm i\frac{V_A\sin\alpha}{\sqrt{\nu\eta}}\left[1+i\frac{1+\mathrm{Pm}}{2S_\eta\sin^2\alpha}+O\left(S_\eta^{-2}\right)\right]=\pm i\frac{V_A\sin\alpha}{\sqrt{\nu\eta}}\left[1+i\frac{\omega(\nu+\eta)}{2V_A^2\sin^2\alpha}+O\left(S_\eta^{-2}\right)\right]
\end{aligned}
$$

exactly the same as the expressions in the first section.

3.4.2 Comparison with numerical solutions

To further verify the result, I include here a comparison between the first order approximation with the numerical solution to eq.(70). In first example, I fix the Lundquist number at $S_\eta = 10^3$, and magnetic Prandtl number at $\mathrm{Pm} = 10^{-3}$; the angle between $\mathbf{B}_0$ and $\hat{\mathbf{x}}$ is fixed at $\alpha = 60°$. The dimensionless horizontal wavenumber $\bar{p}$ varies from 0.1 to ≈ 30. The results are shown in figs.(4) and (5). The first-order approximation works amazingly well, except for a tiny part at relatively high $\bar{p}$ for the damping rate of the travelling Alfvén wave. Such behaviour is possible as $\bar{p}^2 \sim S_\eta$ at $\bar{p} \sim 30$, potentially elevating the higher-order terms to the same level as the first-order correction.

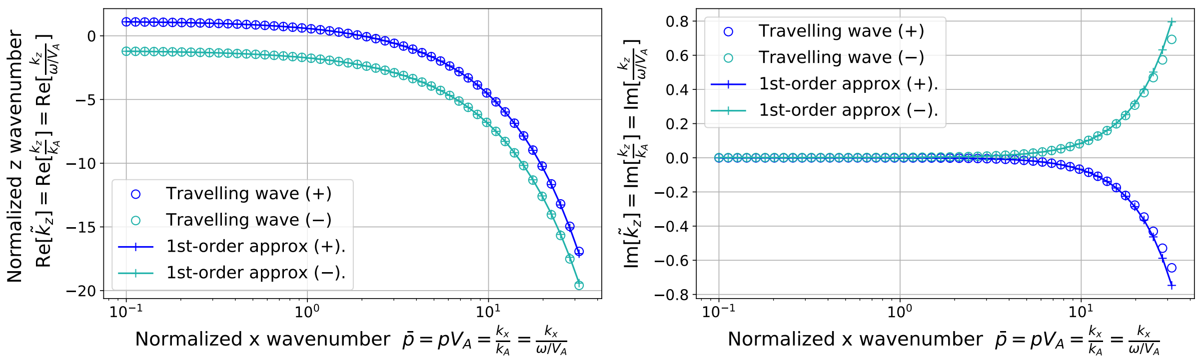

Figure 4: $\tilde{k}_z$ for travelling Alfvén waves as a function of $\bar{p}$, first-order approximation (eq.76) (crossed lines) validated against numerical solutions to eq.(70) (circles).

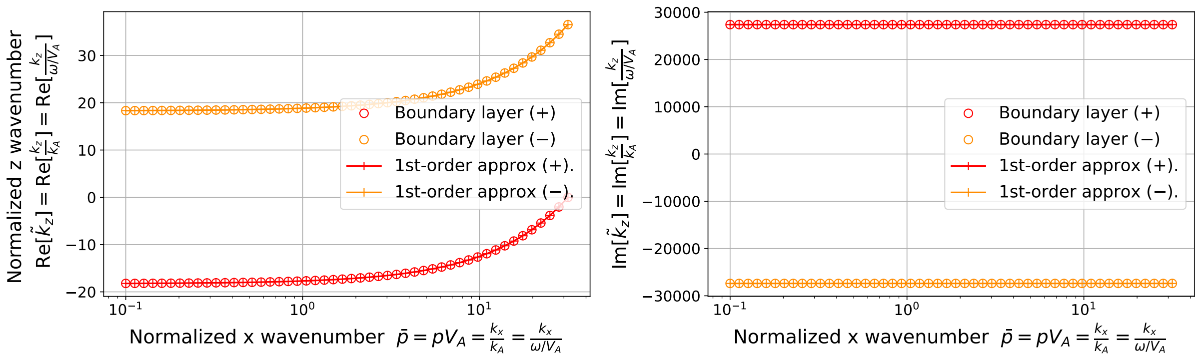

Figure 5: $\tilde{k}_z$ for Hartmann boundary layers as a function of $\bar{p}$, first-order approximation (eq.79) (crossed lines) validated against numerical solutions to eq.(70) (circles).

In the second example, I change the Lundquist number S_η from 1 to 10^5. The rest of the parameters are fixed at $\mathrm{Pm} = 1$, $\bar{p} = 1$ and $\alpha = 60°$. The results are shown in figs. (6) and (7). Significant discrepancy begins to emerge at $S_\eta \sim 1$, but the first-order approximation seems to work well whenever $S_\eta > 30$. I therefore conclude that the first-order approximation developed using pertubative approach is accurate for a wide range of configurations, and can be used in place of numerical solutions for developing analytical reflection and transmission formulae.

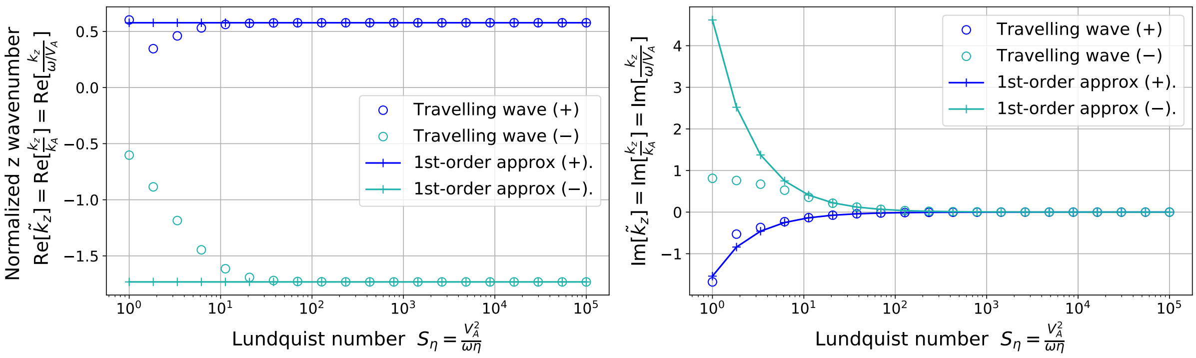

Figure 6: $\tilde{k}_z$ for travelling Alfvén waves as a function of S_η, first-order approximation (eq.76) (crossed lines) validated against numerical solutions to eq.(70) (circles).

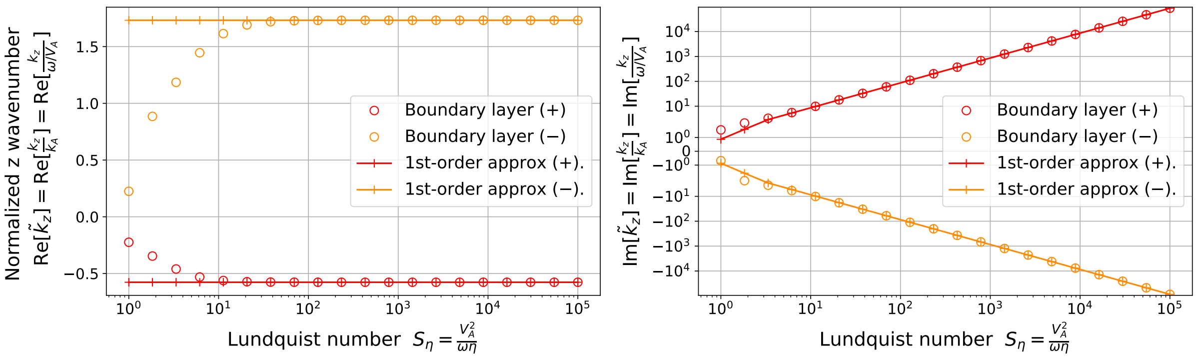

Figure 7: $\tilde{k}_z$ for Hartmann boundary layers as a function of S_η, first-order approximation (eq.79) (crossed lines) validated against numerical solutions to eq.(70) (circles).

4 3-D Reflection at solid boundary

In this section, I shall collect the derivations from previous sections and show how the problem of 3-D reflection and transmission of Alfvén waves can be solved with different levels of approximations.

I shall start by identifying the boundary conditions that need to be satisfied at the fluid-solid interface. As an Alfvén wave impinges on the boundary, the fields at the boundary have the tendency to oscillate together with the incoming wave. These tendencies of oscillation then excite several waves of different nature, that take the energy away from the boundary.

Depending on the specific approximations / assumptions, different waves might not be of equal importance in the system, and might not even be excited in the first place. In the end, the necessary continuity conditions in combination with the relevant waves should form a nonsingular linear system, whose solution gives the reflection and transmission relations.

The general setup is as follows. We look at the scenario where the fluid-solid boundary is at $z = 0$, extending to infinity in the x and y direction. Fluid occupies the $z < 0$ half space, and a solid phase occupies $z > 0$. The fluid is electrically conducting, with finite conductivity. The solid is assumed to be diffusionless with isotropic linear elasticity, at least in the frequency band of interest. The velocities, whether in the fluid or in the solid, are always sufficiently small, that the low-speed limit of either the Galilean transform or the Lorentz transform is satisfactory.

4.1 Identifying the continuity conditions, the fields, and wave components

4.1.1 Continuity conditions at the boundary

First, we look at kinematic boundary conditions. We assume that the two media retain their respective continuum, and have no mixing nor cavity in between. Thus, the non-penetration boundary condition requires

$$[[\hat{\mathbf{n}} \cdot \mathbf{u}]] = \hat{\mathbf{n}} \cdot [[\mathbf{u}]] = [[u_z]] = 0. \tag{80}$$

If the fluid is diffusive, the no-relative-slip boundary condition also needs to be satisfied:

$$[[\hat{\mathbf{n}} \times \mathbf{u}]] = \hat{\mathbf{u}} \times [[\mathbf{u}]] = \mathbf{0}, \quad [[u_x]] = [[u_y]] = 0. \tag{81}$$

When both assumptions are made (as is the case in this section), then the total effect is to impose continuity of the velocity field across the boundary, comprising three scalar equations (u_x, u_y, u_z).

We next look at the dynamic boundary conditions. Discontinuities in stress would give rise to infinitely large accelerations, which should be excluded from the system. Therefore, the traction should be continuous across the boundary,

$$[[\mathbf{t}]] = [[\hat{\mathbf{n}} \cdot \boldsymbol{\tau}]] = \mathbf{0} \tag{82}$$

where $\boldsymbol{\tau}$ is the stress tensor. This comprises another three scalar equations $(\tau_{zz}, \tau_{zx}, \tau_{zy})$.

The aforementioned six boundary conditions are also present in the analysis of elastic waves or acoustic waves. However, the electrically conducting fluid requires additional electromagnetic boundary conditions. As a collorary of Gauss's law for magnetism, the boundary-normal component of the magnetic field must be continuous, giving

$$[[\hat{\mathbf{n}} \cdot \mathbf{B}]] = \hat{\mathbf{n}} \cdot [[\mathbf{B}]] = [[B_z]] = 0. \tag{83}$$

The continuity of the boundary-parallel magnetic field is generally not true in some ideal cases, but remains true when void of sheet currents

$$[[\hat{\mathbf{n}} \times \mathbf{B}]] = \hat{\mathbf{n}} \times [[\mathbf{B}]] = \mathbf{0}, \quad [[B_x]] = [[B_y]] = 0. \tag{84}$$

This is discussed and justified in, e.g. Olson et al. (2015), where Paul Roberts postulated that a thin current sheet should diffuse into currents with finite amplitudes within a finite thickness in the imperfectly conducting fluid (as is the case in this section). Therefore there are another three scalar equation concerning B_x, B_y, B_z.

Although quite often not imposed, the boundary-parallel electric field is required by Faraday's law given finite magnetic field,

$$[[\hat{\mathbf{n}} \times \mathbf{E}]] = \hat{\mathbf{n}} \times [[\mathbf{E}]] = \mathbf{0}, \quad [[E_x]] = [[E_y]] = 0. \tag{85}$$

If one constructs closed curves whose enclosed surface is parallel to the interface, it can be shown that due to Faraday's law

$$\oint_{\partial\Sigma} \mathbf{E} \cdot d\mathbf{l} = -\int_{\Sigma} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S},$$

the two continuities on E_x and E_y and the continuity on B_z are linearly dependent. One can choose either two of the three conditions. Here I shall use the continuity of the boundary-normal magnetic field, plus continuity of one component of the boundary-parallel electric field. The continuity of electric field therefore gives only one additional scalar boundary condition.

To sum up, we have the continuity conditions on the following ten fields

$$u_x, u_y, u_z, \tau_{zx}, \tau_{zy}, \tau_{zz}, b_x, b_y, b_z, e_x(e_y).$$

4.1.2 Coupling of waves with Alfvén waves

Without loss of generality, we look at the problem where the incoming wave is in the Oxz plane. Due to the matching of phase at the boundary, all reflected and transmitted waves should share the same horizontal wavenumber, and hence should be coplanar with the incidence wave. Therefore, Oxz is not only the plane of incidence, but the plane where all wave vectors fall in. The matching of azimuth that Ferraro (1954) comes up with is then a trivial collorary using this parameterization.

Not all Alfvén waves excite all fields at the boundary. For instance, if the incoming wave is polarized purely in the horizontal direction (i.e. in $\hat{\mathbf{y}}$), then the fields u_z, u_x, b_z and b_x are all zero in the interior. Without these oscillations, σ_{zz}, σ_{zx}, e_y are all zero. It follows that when such waves hit the fluid-solid interface, the interface also does not "see" or "feel" any variations in u_z, u_x, b_z, b_x, e_x, σ_{zz} or σ_{zx}. The electromagnetic, elastic, or hydromagnetic waves associated with these fields only will also not be excited.

Table 1: Waves and their relevant fields

Mode	Notation	Kinematic	Dynamic	Magnetic	Electric
Alfvén horizontal	AH	u_y	σ_{zy}	b_y	$e_x, (e_z)$
Alfvén vertical	AV	u_x, u_z	σ_{zx}, σ_{zz}	b_x, b_z	e_y
Hartmann horizontal	BLH	u_y	σ_{zy}	b_y	$e_x, (e_z)$
Hartmann vertical	BLV	u_x, u_z	σ_{zx}, σ_{zz}	b_x, b_z	e_y
S-wave horizontal	SH	u_y	σ_{zy}	*	*
S-wave vertical	SV	u_x, u_z	σ_{zx}, σ_{zz}		
P/Acoustic	P	u_x, u_z	σ_{zx}, σ_{zz}		
Horizontal magnetic	MH	*	*	b_y	$e_x, (e_z)$
Horizontal electric	EH			b_x, b_z	e_y

The physical fields associated with these waves are summarized in Table 1. The bracket around e_z indicates although this field is excited, there is usually no boundary condition imposed on it. Discontinuities in e_z is generally matched with a sheet of electric charge at the boundary.

Similar to the case with elastic waves, it seems good pratice to separate the Alfvén waves into horizontally-polarized (AH) and vertically-polarized (AV) modes. These two modes excite two distinct, mutually-exclusive sets of fields, and assuming that the elastic/electromagnetic waves have no electromagnetic/mechanical effects, respectively, the two polarization of Alfvén waves also only couple with selected elastic/electromagnetic waves. For instance, AH mode is coupled with SH, MH modes, while AV is coupled with P, SV and EH modes.

An open question here is whether elastic/electromagnetic waves in the medium are purely mechanical/electromagnetic, or also have electromagnetic/mechanical effects. This ambiguity is marked with asteriks in the table. [While it seems plausible and intuitive that these waves may have marginal electromagnetic/mechanical effects, it remains uncertain what is the error made in this assumption (or what is the dimensionless number that gives the amplitude (energy) partition ratio between velocity field (kinematic energy) and electromagnetic field (electromagnetic energy)).] Here I shall assume that the effect is marginal, and the asterik in the table indicates no such fields excited. I shall come back to this assumption at the end of the section.

4.1.3 Two modes and related conditions

I have already shown that, assuming mechanical waves and electromagnetic waves are purely mechanical and electromagnetic, respectively, AH and AV modes are only coupled with selected modes. In this part, I shall list the coupled waves and their relevant boundary conditions.

First, we collect all waves that are involved in the fields u_y, σ_{zy}, b_y and e_x. Counting both incidental and reflected Alfvén waves, we have altogether five components:

- (Incidence) horizontally-polarized Alfvén wave (AH-Incidence),
- (Reflected) horizontally-polarized Alfvén wave (AH-Reflected),
- Horizontally-polarized Hartmann boundary layer (BLH),
- (Transmitted) horizontally-polarized longitudinal wave (SH-Transmitted),
- (Transmitted) electromagnetic wave with horizontal magnetic field (MH-Transmitted).

The five components are linked via four continuity conditions:

$$[[u_y]] = 0, \quad [[\sigma_{zy}]] = 0, \quad [[b_y]] = 0, \quad [[e_z]] = 0, \tag{86}$$

which should yield four reflection/refraction coefficients.

The second scenario couples waves involving fields $u_x, u_z, \sigma_{zx}, \sigma_{zx}, b_x, b_z$. Counting both incidental and reflected Alfvén waves, we have altogether seven components:

- (Incidence) vertically-polarized Alfvén wave (AV-Incidence),
- (Reflected) vertically-polarized Alfvén wave (AV-Reflected),
- Vertically-polarized Hartmann boundary layer (BLV),
- (Reflected) acoustic wave (P-Reflected),
- (Reflected) compressional wave (P-Transmitted)
- (Transmitted) vertically-polarized longitudinal wave (SV-Transmitted),
- (Transmitted) electromagnetic wave with horizontal electric field (EH-Transmitted).

The six continuity conditions involved are

$$[[u_x]] = [[u_z]] = 0, \quad [[\sigma_{zy}]] = [[\sigma_{zz}]] = 0, \quad [[b_x]] = [[b_z]] = 0, \tag{87}$$

No additional condition is required for e_y, as $e_x = 0$, and continuity of b_z would entail continuity of e_y.

4.2 Waves and associated fields

In this subsection, I shall summarize the plane wave solutions of different waves. Most of the content will be reiteration of what has already been derived in previous sections, but I shall list the relation between different fields involved in matrix notations.

The idea of these concise matrix-vector notations are inspired by Aki and Richards (2002). In their treatment of seismic waves at medium interfaces, Aki and Richards (2002) used these notations to match the boundary conditions, something I shall also do here. They also showed that by writing the waves in a general form $\frac{d}{dz}\mathbf{f} = \mathbf{A}\mathbf{f}$, different modes/waves that are permitted in the system can be solved as eigenvectors of $\mathbf{A}$, with their respective vertical slowness as eigenvalues. This is another, perhaps more systematic way to derive the dispersion relation for k_z. Although methodologically elegant, for anisotropic waves such as Alfvén waves, this approach proves to yield systems that are no simpler than the original form. I walked halfway through this approach, but decided in the end I shall simply reuse the results from previous sections.

4.2.1 The Alfvén waves

For the travelling Alfvén waves, I shall start from the previously derived zeroth-order approximate dispersion relation. Assuming a horizontal slowness denoted by p, the plane wave ansatz takes the form

$$\mathbf{u} \exp\{i(\omega t - k_x x - k_z z)\} = \mathbf{u} \exp\{i(\omega t - \omega p x - k_z z)\}.$$

The other three fields are related to the velocity field via

$$\begin{aligned} \frac{\partial \mathbf{b}}{\partial t} &= \mathbf{B}_0 \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{b} \\ \boldsymbol{\tau} &= \rho\nu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) + \frac{\mathbf{B}_0 \cdot \mathbf{b}}{\mu_0}\mathbf{I} \\ \mathbf{e} &= \eta \nabla \times \mathbf{b} - \mathbf{u} \times \mathbf{B}_0 \end{aligned} \tag{88}$$

which can be rewritten in the frequency-wavenumber domain (using the plane wave ansatz)

$$\begin{aligned} \mathbf{b} &= \frac{-iB_x k_x - iB_z k_z}{i\omega + \eta(k_x^2 + k_z^2)}\mathbf{u} = -\sqrt{\rho\mu_0} V_A \frac{k_x \cos\alpha + k_z \sin\alpha}{\omega - i\eta(k_x^2 + k_z^2)}\mathbf{u} = \sqrt{\rho\mu_0} A_k \mathbf{u} \\ \boldsymbol{\tau} &= \rho\nu \left[-ik_x(\hat{\mathbf{x}}\mathbf{u} + \mathbf{u}\hat{\mathbf{x}}) - ik_z(\hat{\mathbf{z}}\mathbf{u} + \mathbf{u}\hat{\mathbf{z}})\right] + \rho A_k V_A \left[(\cos\alpha\hat{\mathbf{x}} + \sin\alpha\hat{\mathbf{z}}) \cdot \mathbf{u}\right]\mathbf{I} \\ \mathbf{e} &= -\eta(ik_x\hat{\mathbf{x}} + ik_z\hat{\mathbf{z}}) \times (\sqrt{\rho\mu_0} A_k \mathbf{u}) + B_0(\cos\alpha\hat{\mathbf{x}} + \sin\alpha\hat{\mathbf{z}}) \times \mathbf{u} \\ &= -i\sqrt{\rho\mu_0}\eta A_k (k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}) \times \mathbf{u} + \sqrt{\rho\mu_0} V_A (\cos\alpha\hat{\mathbf{x}} + \sin\alpha\hat{\mathbf{z}}) \times \mathbf{u} \end{aligned} \tag{89}$$

where the dimensionless number A_k gives the ratio between magnetic and velocity fields

$$A_k = -\frac{V_A(k_x \cos\alpha + k_z \sin\alpha)}{\omega - i\eta(k_x^2 + k_z^2)}. \tag{90}$$

Note that the spatially varying magnetic pressure also contributes an isotropic component to the total stress. This comes from the balance $\nabla(p + p^M) = \mathbf{0}$. Observing that a constant factor of $\sqrt{\rho\mu_0}$ is required to bridge the gap between kinematic and electromagnetic fields, we redefine

$$\mathbf{b} := \frac{\mathbf{b}}{\sqrt{\rho\mu_0}}, \quad \mathbf{e} := \frac{\mathbf{e}}{\sqrt{\rho\mu_0}}. \tag{91}$$

Under these transforms, $\mathbf{b}$ has the a dimension of velocity, and $[\boldsymbol{\tau}] = [\rho][\mathbf{e}]$. I shall use this redefinition for the rest of this section. The relations between the fields are then rewritten as

$$\begin{aligned} \mathbf{b} &= A_k \mathbf{u} \\ \boldsymbol{\tau} &= -i\rho\nu \left[k_x(\hat{\mathbf{x}}\mathbf{u} + \mathbf{u}\hat{\mathbf{x}}) + k_z(\hat{\mathbf{z}}\mathbf{u} + \mathbf{u}\hat{\mathbf{z}})\right] + \rho A_k V_A \left[(\cos\alpha\hat{\mathbf{x}} + \sin\alpha\hat{\mathbf{z}}) \cdot \mathbf{u}\right]\mathbf{I} \\ \mathbf{e} &= -i\eta A_k (k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}) \times \mathbf{u} + V_A(\cos\alpha\hat{\mathbf{x}} + \sin\alpha\hat{\mathbf{z}}) \times \mathbf{u} \\ &= (-i\eta A_k k_x + V_A \cos\alpha)\hat{\mathbf{x}} \times \mathbf{u} + (-i\eta A_k k_z + V_A \sin\alpha)\hat{\mathbf{z}} \times \mathbf{u}. \end{aligned} \tag{92}$$

We have seen that the z-wavenumber for the travelling Alfvén wave is given by (eq.??)

$$k_z = \frac{\omega}{V_A}\frac{\pm 1 - pV_A\cos\alpha}{\sin\alpha} + O(\epsilon_\eta).$$

Once again $\epsilon_\eta = 1/S_\eta = \omega\eta/V_A^2$ is the inverse of the Lundquist number. This relation has a relative error of $O(\epsilon_\eta)$, meaning the error in k_z is at the order of $\frac{\omega}{V_A}\epsilon_\eta$. It follows that

$$\begin{aligned} V_A(k_x\cos\alpha + k_z\sin\alpha) &= \omega p V_A\cos\alpha + k_z V_A\sin\alpha = \pm\omega + O(\epsilon_\eta) \\ k_x^2 + k_z^2 = \omega^2 p^2 + k_z^2 &= \frac{\omega^2}{V_A^2}\left(\left(1 + p^2V_A^2\right)\csc^2\alpha \mp pV_A\csc\alpha\cot\alpha\right) + O(\epsilon_\eta) \end{aligned} \tag{93}$$

For moderate incidence angle, $pV_A = \bar{p} <\sim 1$ and $\sin\alpha \sim 1$, the factor

$$\begin{aligned} A_k &= -\frac{V_A(k_x\cos\alpha + k_z\sin\alpha)}{\omega - i\eta(k_x^2 + k_z^2)} \\ &= \mp\frac{\omega + O(\epsilon_\eta)}{\omega - i\frac{\omega^2\eta}{V_A^2}\left((1+\bar{p}^2)\csc^2\alpha \mp \bar{p}\csc\alpha\cot\alpha + O(\epsilon_\eta)\right)} \\ &= \mp\frac{1 + O(\epsilon_\eta)}{1 - i\epsilon_\eta\left((1+\bar{p}^2)\csc^2\alpha \mp \bar{p}\csc\alpha\cot\alpha + O(\epsilon_\eta)\right)} \\ &= \mp 1 + O(\epsilon_\eta). \end{aligned} \tag{94}$$

Plugging this into the relation of fields, we have for the travelling Alfvén wave,

$$\begin{aligned} \mathbf{b} &= \mp\left(1 + O(\epsilon_\eta)\right)\mathbf{u} \\ \boldsymbol{\tau} &= -i\rho\nu[k_x(\hat{\mathbf{x}}\mathbf{u} + \mathbf{u}\hat{\mathbf{x}}) + k_z(\hat{\mathbf{z}}\mathbf{u} + \mathbf{u}\hat{\mathbf{z}})] \mp \rho V_A(\cos\alpha\hat{\mathbf{x}} + \sin\alpha\hat{\mathbf{z}})\cdot\mathbf{u}\mathbf{I} \\ &= -i\rho\frac{\omega\nu}{V_A}\left[pV_A(\hat{\mathbf{x}}\mathbf{u} + \mathbf{u}\hat{\mathbf{x}}) + \frac{\pm 1 - pV_A\cos\alpha}{\sin\alpha}(\hat{\mathbf{z}}\mathbf{u} + \mathbf{u}\hat{\mathbf{z}})\right] \mp \rho V_A(\cos\alpha\hat{\mathbf{x}} + \sin\alpha\hat{\mathbf{z}})\cdot\mathbf{u}\mathbf{I} \\ \mathbf{e} &= \left(\pm i\eta k_x(1 + O(\epsilon_\eta)) + V_A\cos\alpha\right)\hat{\mathbf{x}}\times\mathbf{u} + \left(\pm i\eta k_z(1 + O(\epsilon_\eta)) + V_A\sin\alpha\right)\hat{\mathbf{z}}\times\mathbf{u} \\ &= \left(\cos\alpha \pm i\frac{\omega\eta}{V_A^2}pV_A(1 + O(\epsilon_\eta))\right)V_A\hat{\mathbf{x}}\times\mathbf{u} + \left(\sin\alpha \pm i\frac{\omega\eta}{V_A^2}\frac{\pm 1 - pV_A\cos\alpha}{\sin\alpha}\right)V_A\hat{\mathbf{z}}\times\mathbf{u} \\ &= \left(\cos\alpha + O(\epsilon_\eta)\right)V_A\hat{\mathbf{x}}\times\mathbf{u} + \left(\sin\alpha + O(\epsilon_\eta)\right)V_A\hat{\mathbf{z}}\times\mathbf{u}. \end{aligned} \tag{95}$$

In the original dispersion relation of k_z, the positive and negative signs correspond to waves travelling in the $+z$ and $-z$ directions, respectively. Therefore, in eq.(95), the upper and lower signs always correspond to waves propagating in the $+z$ and $-z$ directions, respectively. We also observe that when $\sin\alpha$ is of order unity, the contribution of $\eta\nabla\times\mathbf{b}$ in the electric field is of order ϵ_η compared to the motion-induced field $\mathbf{u}\times\mathbf{B}_0$. The dominant component in electromagnetic fields as well as velocity fields are the same as in the ideal limit.

We are now ready to derive the fields related to the two Alfvén wave modes. For the horizontally-polarized Alfvén wave (AH), we can set up the field vector as

$$\mathbf{f}^{\mathrm{AH}} = \begin{pmatrix} u_y \\ \tau_{zy} \\ b_y \\ e_x \end{pmatrix} = \begin{pmatrix} 1 \\ -i\frac{\rho\omega\nu}{V_A}\frac{\pm 1 - pV_A\cos\alpha}{\sin\alpha} \\ \mp 1 \\ -V_A\sin\alpha \end{pmatrix} u_y \tag{96}$$

In each of the fields, only terms up to the leading order of ϵ_η are collected. The elements in the vector are collected from the corresponding components in eq.(95). For the vertically-polarized Alfvén wave, the problem is more complicated. Not only are there six scalar field components involved, but the polarization is no longer along the principal axes. Using the rotation transform matrix is of course possible, but even the velocity and magnetic fields would be quite complex in the $Oxyz$ components, let alone the secondary fields. Instead, I define the stream function $\mathbf{\Psi} = \psi\hat{\mathbf{y}}$ so that

$$\mathbf{u} = \nabla \times \Psi, \quad u_x = -\frac{\partial\psi}{\partial z} = ik_z\psi, \quad u_z = \frac{\partial\psi}{\partial x} = -ik_x\psi. \tag{97}$$

The field vector for the vertically-polarized Alfvén wave (AV) is then given by

$$\mathbf{f}^{\mathrm{AV}} = \begin{pmatrix} u_x \\ u_z \\ \tau_{zx} \\ \tau_{zz} \\ b_x \\ b_z \end{pmatrix} = \begin{pmatrix} ik_z \\ -ik_x \\ \rho\nu\left(k_z^2 - k_x^2\right) \\ -2\rho\nu k_x k_z \mp i\rho V_A(k_z\cos\alpha - k_x\sin\alpha) \\ \mp ik_z \\ \pm ik_x \end{pmatrix}\psi$$
$$= \begin{pmatrix} i\frac{\omega}{V_A}\left(\pm\csc\alpha - \bar{p}\cot\alpha\right) \\ -i\omega p \\ \frac{\rho\nu\omega^2}{V_A^2}\left(\csc^2\alpha + \bar{p}^2(\cot^2\alpha - 1) \mp 2\bar{p}\cot\alpha\csc\alpha\right) \\ -2\frac{\rho\nu\omega^2}{V_A^2}\bar{p}\left(\pm\csc\alpha - \bar{p}\cot\alpha\right) + i\rho\omega\frac{-\cos\alpha \pm pV_A}{\sin\alpha} \\ i\frac{\omega}{V_A}\left(-\csc\alpha \pm \bar{p}\cot\alpha\right) \\ \pm i\omega p \end{pmatrix}\psi \tag{98}$$

4.2.2 The Hartmann layer

The Hartmann layer is derived from the same set of equations as the Alfvén waves, hence the equations that govern the relations between fields are the same (eq.92). One only needs to plug in a different dispersion relation (eq.**??**)

$$k_z = \pm i\frac{\omega}{V_A}\frac{S_\eta \sin\alpha}{\sqrt{\mathrm{Pm}}} + O(\epsilon_\eta) = \pm i\frac{V_A\sin\alpha}{\sqrt{\nu\eta}} + O(\epsilon_\eta).$$

The only allowed solution here is $\mathrm{Im}[k_z] > 0$. This solution gives a wave that is decaying in the $-z$ direction, which is permitted in the system. The quantities

$$\begin{aligned} \omega p V_A\cos\alpha + k_z V_A\sin\alpha &= \frac{V_A^2}{\sqrt{\nu\eta}}\left(\frac{\omega\sqrt{\nu\eta}}{V_A^2}pV_A\cos\alpha + i\sin^2\alpha\right) = i\frac{V_A^2}{\sqrt{\nu\eta}}\sin^2\alpha + O(\epsilon_\eta) \\ \omega^2p^2 + k_z^2 &= \frac{V_A^2}{\nu\eta}\left(\left(\frac{\omega\sqrt{\nu\eta}}{V_A^2}pV_A\right)^2 - \sin^2\alpha + O(\epsilon_\eta)\right) = -\frac{V_A^2}{\nu\eta}\sin^2\alpha + O(\epsilon_\eta) \end{aligned} \tag{99}$$

are all dominated by the imaginary k_z with large absolute value to first order. Therefore, we have

$$\begin{aligned} A_k &= -\frac{V_A(k_x\cos\alpha + k_z\sin\alpha)}{\omega - i\eta(k_x^2 + k_z^2)} = -\frac{iV_A^2\sin^2\alpha/\sqrt{\nu\eta} + O(\epsilon_\eta)}{iV_A^2\sin^2\alpha/\nu\left(1 - i\frac{\omega\nu}{V_A^2}\csc^2\alpha + O(\epsilon_\eta)\right)} \\ &= -\sqrt{\frac{\nu}{\eta}} + O(\epsilon_\eta) = -\sqrt{\mathrm{Pm}} + O(\epsilon_\eta) \end{aligned} \tag{100}$$

In the end, using these quantities, eq.(92) is rewritten as

$$\begin{aligned} \mathbf{b} &= -\sqrt{\mathrm{Pm}}\,\mathbf{u} \\ \boldsymbol{\tau} &= -i\rho\nu\omega p(\hat{\mathbf{x}}\mathbf{u} + \mathbf{u}\hat{\mathbf{x}}) + \rho\nu\frac{V_A\sin\alpha}{\sqrt{\nu\eta}}(\hat{\mathbf{z}}\mathbf{u} + \mathbf{u}\hat{\mathbf{z}}) - \sqrt{\mathrm{Pm}}\rho V_A[(\cos\alpha\hat{\mathbf{x}} + \sin\alpha\hat{\mathbf{z}})\cdot\mathbf{u}]\mathbf{I} \\ \mathbf{e} &= \left(i\sqrt{\nu\eta}\omega p + V_A\cos\alpha\right)\hat{\mathbf{x}}\times\mathbf{u} + O(\epsilon_\eta)V_A\sin\alpha\hat{\mathbf{z}}\times\mathbf{u}. \end{aligned} \tag{101}$$

The field vector encoding the Hartmann boundary layer polarized along y axis can be written as

$$\mathbf{f}^{\mathrm{BLH}} = \begin{pmatrix} u_y \\ \tau_{zy} \\ b_y \\ e_x \end{pmatrix} = \begin{pmatrix} 1 \\ \rho\sqrt{\mathrm{Pm}}V_A \sin\alpha \\ -\sqrt{\mathrm{Pm}} \\ 0 \end{pmatrix} u_y. \tag{102}$$

A surprising revelation is that such Hartmann layer solution is void of electric field e_x, at least to the leading order of ϵ_η, relative to $V_A u_y$. The electric field, that gives rise to the conducted current, cancels out in the Galilean transform with the term $\mathbf{u}\times\mathbf{B}_0$. Once again employing the y-component of the stream function, the in-plane mode of Hartmann layer is encoded by the vector

$$\begin{aligned}
\mathbf{f}^{\mathrm{BLV}} = \begin{pmatrix} u_x \\ u_z \\ \tau_{zx} \\ \tau_{zz} \\ b_x \\ b_z \end{pmatrix} &= \begin{pmatrix} ik_z \\ -ik_x \\ \rho\nu(k_z^2 - k_x^2) \\ -2\rho\nu k_x k_z + i\rho A_k V_A(k_z\cos\alpha - k_x\sin\alpha) \\ iA_k k_z \\ -iA_k k_x \end{pmatrix}\psi \\
&= \begin{pmatrix} -\frac{V_A\sin\alpha}{\sqrt{\nu\eta}} \\ -i\omega p \\ -\rho\frac{V_A^2}{\eta}\left(\sin\alpha^2 + \bar{p}^2\left(\frac{\omega\eta}{V_A^2}\right)^2\frac{\mathrm{Pm}}{\sin^2\alpha}\right) \\ -2i\sqrt{\mathrm{Pm}}\rho\omega\bar{p}\sin\alpha + \rho\omega\sin\alpha\left(\frac{V_A^2}{\omega\eta}\cos\alpha + i\bar{p}\sqrt{\mathrm{Pm}}\right) \\ \frac{V_A\sin\alpha}{\eta} \\ i\sqrt{\mathrm{Pm}}\omega p \end{pmatrix}\psi \\
&\approx \begin{pmatrix} -\frac{V_A}{\sqrt{\nu\eta}}\sin\alpha \\ -i\omega p \\ -\rho\frac{V_A^2}{\eta}\sin^2\alpha \\ -i\rho\omega p V_A\sin\alpha\sqrt{\mathrm{Pm}} + \frac{\rho V_A^2}{\eta}\sin\alpha\cos\alpha \\ \frac{V_A}{\eta}\sin\alpha \\ i\omega p\sqrt{\mathrm{Pm}} \end{pmatrix}\psi.
\end{aligned} \tag{103}$$

4.2.3 The acoustic/elastic waves

In the derivations of Alfvén waves in the first section, I have made the assumption of incompressible fluid. This is justifiable since the typical velocities both of Alfvén wave and of the fluid parcels are much smaller than the sound speed. However, I have to relax this approximation because of the additional continuity conditions.

[The coexistence of Alfvén wave and acoustic wave in a compressible medium is shown in the appendix]. In an elastic medium, a longitudinal wave (S-wave) is further possible. The velocities of compressional wave (P) and longitudinal wave (S) are given by α and β, respectively. In the fluid, $\beta \longrightarrow 0$.

In this part I shall assume that the acoustic wave has negligible electromagnetic effects, an assumption that will be looked at in more details later. Under this assumption, these waves are simply pure acoustic/elastic waves in isotropic medium. The plane wave ansatz gives

$$\mathbf{u}^P = \mathbf{u}^P \exp\left\{i\omega\left(t - px \mp \sqrt{\frac{1}{\alpha^2} - p^2}z\right)\right\}, \quad \mathbf{u}^S = \mathbf{u}^S \exp\left\{i\omega\left(t - px \mp \sqrt{\frac{1}{\beta^2} - p^2}z\right)\right\}. \tag{104}$$

Note these are velocity fields, same as in fluids. The associated displacement fields requires a time integral, and takes the form

$$\mathbf{U}^P = \int \mathbf{u}^P \, dt = \frac{\mathbf{u}^P}{i\omega}, \quad \mathbf{U}^S = \int \mathbf{u}^S \, dt = \frac{\mathbf{u}^S}{i\omega}. \tag{105}$$

The stress fields are given by the constitutive relation

$$\begin{aligned}\boldsymbol{\tau} &= \lambda(\nabla\cdot\mathbf{U})\mathbf{I} + \mu(\nabla\mathbf{U} + \nabla\mathbf{U}^T) \\ &= \frac{\lambda}{i\omega}(\nabla\cdot\mathbf{u})\mathbf{I} + \frac{\mu}{i\omega}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) \\ &= -\frac{\lambda}{\omega}\left(k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}\right)\cdot\mathbf{u}\,\mathbf{I} - \frac{\mu}{\omega}\left[k_x(\hat{\mathbf{x}}\mathbf{u} + \mathbf{u}\hat{\mathbf{x}}) + k_z(\hat{\mathbf{z}}\mathbf{u} + \mathbf{u}\hat{\mathbf{z}})\right]\end{aligned} \tag{106}$$

where λ and μ are Lame constants, linked to wave velocities via

$$\alpha^2 = \frac{\lambda + 2\mu}{\rho}, \quad \beta^2 = \frac{\mu}{\rho}.$$

I shall separate the acoustic/elastic waves into three modes. For SH wave in elastic medium, $\mathbf{u} = u_y\hat{\mathbf{y}}$. The stress is given by

$$\begin{aligned}\boldsymbol{\tau} &= -\mu\left[p(\hat{\mathbf{x}}\hat{\mathbf{y}} + \hat{\mathbf{y}}\hat{\mathbf{x}}) \pm \sqrt{\frac{1}{\beta^2} - p^2}(\hat{\mathbf{z}}\hat{\mathbf{y}} + \hat{\mathbf{y}}\hat{\mathbf{z}})\right]u_y \\ &= -\rho\beta^2 p(\hat{\mathbf{x}}\hat{\mathbf{y}} + \hat{\mathbf{y}}\hat{\mathbf{x}})u_y \mp \rho\beta\sqrt{1 - p^2\beta^2}(\hat{\mathbf{z}}\hat{\mathbf{y}} + \hat{\mathbf{y}}\hat{\mathbf{z}})u_y.\end{aligned} \tag{107}$$

SH wave only generates shear stress between xy and zy. In particular, the τ_{zy} component couples this mode with similarly horizontally polarized Alfvén waves and Hartmann layer solutions. The SH wave is encoded by the field vector

$$\mathbf{f}^{\mathrm{SH}} = \begin{pmatrix} u_y \\ \tau_{zy} \\ b_y \\ e_x \end{pmatrix} = \begin{pmatrix} 1 \\ \mp\rho\beta\sqrt{1 - p^2\beta^2} \\ 0 \\ 0 \end{pmatrix} u_y. \tag{108}$$

For P-wave in elastic medium, or acoustic wave in the fluid, I again invoke the scalar potential ϕ, so that $u_x = \partial_x\phi = -i\omega p\phi$ and $u_z = \partial_z\phi = \mp i\omega\sqrt{\alpha^{-2} - p^2}\phi$. The stress field is related to the velocity field by

$$\begin{aligned}\boldsymbol{\tau} &= \frac{\lambda}{i\omega}\nabla^2\phi\mathbf{I} + 2\frac{\mu}{i\omega}\nabla\nabla\phi = i\lambda\frac{\omega}{\alpha^2}\phi\mathbf{I} + 2i\mu\omega\left(p^2\hat{\mathbf{x}}\hat{\mathbf{x}} + (\alpha^{-2} - p^2)\hat{\mathbf{z}}\hat{\mathbf{z}} \pm p\sqrt{\alpha^{-2} - p^2}(\hat{\mathbf{x}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\mathbf{x}})\right)\phi \\ &= i\rho\omega\left(1 - \frac{\beta^2}{\alpha^2}\right)\phi\mathbf{I} + i2\rho\omega\left(p^2\beta^2\hat{\mathbf{x}}\hat{\mathbf{x}} + \left(\frac{\beta^2}{\alpha^2} - p^2\beta^2\right)\hat{\mathbf{z}}\hat{\mathbf{z}} \pm p\beta\sqrt{\frac{\beta^2}{\alpha^2} - p^2\beta^2}(\hat{\mathbf{x}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\mathbf{x}})\right)\phi\end{aligned} \tag{109}$$

and the P-wave is encoded by

$$\mathbf{f}^{\mathrm{P}} = \begin{pmatrix} b_x \\ b_z \\ u_x \\ u_z \\ \tau_{zx} \\ \tau_{zz} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -i\omega p \\ \mp i\frac{\omega}{\alpha}\sqrt{1 - p^2\alpha^2} \\ \pm 2i\rho\omega p\beta\sqrt{\frac{\beta^2}{\alpha^2} - p^2\beta^2} \\ i\rho\omega\left(1 - 2\frac{\beta^2}{\alpha^2}\right) + 2i\rho\omega\left(\frac{\beta^2}{\alpha^2} - p^2\beta^2\right) \end{pmatrix}\phi = \begin{pmatrix} 0 \\ 0 \\ -i\omega p \\ \mp i\frac{\omega}{\alpha}\sqrt{1 - p^2\alpha^2} \\ \pm i2\rho\omega p\beta\sqrt{\frac{\beta^2}{\alpha^2} - p^2\beta^2} \\ i\rho\omega\left(1 - 2p^2\beta^2\right) \end{pmatrix}\phi. \tag{110}$$

We also note the conversion between the scalar potential and the velocity amplitude

$$u^P = \sqrt{u_x^2 + u_z^2} = -i\frac{\omega}{\alpha}\phi.$$

Using this relation, the P-wave can also be expressed as

$$\mathbf{f}^{\mathrm{P}} = \begin{pmatrix} 0 \\ 0 \\ p\alpha \\ \pm\sqrt{1-p^2\alpha^2} \\ \mp 2\rho p\alpha\beta\sqrt{\frac{\beta^2}{\alpha^2}-p^2\beta^2} \\ -\rho\alpha(1-2p^2\beta^2) \end{pmatrix} u^P. \tag{111}$$

Finally, the SV-wave can be described by a scalar stream function as $\mathbf{u} = \nabla \times (\psi\hat{\mathbf{y}})$. The velocity field thus takes the form $u_x = -\partial_z\psi = ik_z\psi = \pm i\omega\sqrt{\beta^{-2}-p^2}\psi$ and $u_z = \partial_x\psi = -ik_x\psi = -i\omega p\psi$. If we take $u_z/u^{\mathrm{SV}} > 0$ as the positive direction, the amplitude of the wave is expressed in the stream function as

$$u^{\mathrm{SV}} = \sqrt{u_x^2 + u_z^2} = -i\frac{\omega}{\beta}\psi.$$

The stress field is given by

$$\begin{aligned} \boldsymbol{\tau} &= \frac{\mu}{i\omega}\left(-2\frac{\partial^2\psi}{\partial x\partial z}\hat{\mathbf{x}}\hat{\mathbf{x}} + 2\frac{\partial^2\psi}{\partial x\partial z}\hat{\mathbf{z}}\hat{\mathbf{z}} + \left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}\right)(\hat{\mathbf{x}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\mathbf{x}})\right) \\ &= \frac{\mu}{i\omega}\left(2k_xk_z(\hat{\mathbf{x}}\hat{\mathbf{x}} - \hat{\mathbf{z}}\hat{\mathbf{z}}) + (k_z^2 - k_x^2)(\hat{\mathbf{x}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\mathbf{x}})\right)\psi \\ &= \frac{\mu}{i\omega}\left(\pm 2\omega^2 p\sqrt{\frac{1}{\beta^2} - p^2}(\hat{\mathbf{x}}\hat{\mathbf{x}} - \hat{\mathbf{z}}\hat{\mathbf{z}}) + \omega^2\left(\frac{1}{\beta^2} - 2p^2\right)(\hat{\mathbf{x}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\mathbf{x}})\right)\psi \\ &= \mp i2\rho\omega p\beta\sqrt{1-p^2\beta^2}\psi(\hat{\mathbf{x}}\hat{\mathbf{x}} - \hat{\mathbf{z}}\hat{\mathbf{z}}) - i\rho\omega(1-2p^2\beta^2)\psi(\hat{\mathbf{x}}\hat{\mathbf{z}} + \hat{\mathbf{z}}\hat{\mathbf{x}}) \end{aligned} \tag{112}$$

So the SV-wave is encoded by

$$\mathbf{f}^{\mathrm{SV}} = \begin{pmatrix} b_x \\ b_z \\ u_x \\ u_z \\ \tau_{zx} \\ \tau_{zz} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm i\frac{\omega}{\beta}\sqrt{1-p^2\beta^2} \\ -i\omega p \\ -i\rho\omega(1-2p^2\beta^2) \\ \pm i2\rho\omega p\beta\sqrt{1-p^2\beta^2} \end{pmatrix}\psi = \begin{pmatrix} 0 \\ 0 \\ \mp\sqrt{1-p^2\beta^2} \\ p\beta \\ \rho\beta(1-2p^2\beta^2) \\ \mp\rho\beta^2\sqrt{1-p^2\beta^2} \end{pmatrix} u^{\mathrm{SV}}. \tag{113}$$

4.2.4 Electromagnetic waves

Assuming purely electromagnetic waves in the (solid) medium, we have the Ampere's law,

$$\nabla \times \mathbf{b} = \frac{1}{\eta}\mathbf{e} + \frac{1}{c^2}\frac{\partial \mathbf{e}}{\partial t}$$

which reads in the frequency-wavenumber domain

$$\left(\frac{1}{\eta} + i\frac{\omega}{c^2}\right)\mathbf{e} = -i(k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}) \times \mathbf{b}. \tag{114}$$

We always assume that $\eta > \epsilon > 0$ in the solid. When η is finite, i.e. electrically conducting, and $\omega\eta \ll c^2$ or $\sigma \gg \omega\varepsilon_0$, we can safely drop the displacement current and write

$$\mathbf{e} = -i\eta(k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}) \times \mathbf{b}. \tag{115}$$

This is sometimes referred to as the "good conductor" approximation. The solution for k_z under this approximation is given by eq.(59). If such approximation does not hold, e.g. when $\eta \to +\infty$ and $\eta^{-1} \to 0$ as in the insulating medium, the electric field is

$$\mathbf{e} = -\frac{c^2}{\omega}(k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}) \times \mathbf{b}. \tag{116}$$

The solution for k_z under this approximation is given by eq.(64-65). In general, the electric field for finite η which does not fulfill good conductor nor perfect insulator approximations is given by

$$\mathbf{e} = -i\frac{\eta}{1 + i\frac{\omega\eta}{c^2}}(k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}) \times \mathbf{b}. \tag{117}$$

For electromagnetic waves with magnetic field polarized in the y direction, the vector encoding is

$$\mathbf{f}^{\mathrm{MH}} = \begin{pmatrix} u_y \\ \tau_{zy} \\ b_y \\ e_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{i\eta k_z}{1+i\omega\eta/c^2} \end{pmatrix} b_y. \tag{118}$$

When magnetic field is polarized in the Oxz plane, we resort to the y-component of the scalar potential $A = \hat{\mathbf{y}} \cdot \mathbf{A}$ so that $b_x = -\partial_z A = ik_z A$ and $b_z = \partial_x A = -ik_x A$. The vector encoding is

$$\mathbf{f}^{\mathrm{MV}} = \begin{pmatrix} u_x \\ u_z \\ \tau_{zx} \\ \tau_{zz} \\ b_x \\ b_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ ik_z \\ -ik_x \end{pmatrix} A. \tag{119}$$

4.3 Solutions at insulating wall

I state the general assumptions made in this section as follows. The Lundquist number is large ($\epsilon_\eta \ll 1$) although the fluid is not ideal, the magnetic Prandtl number is either of order unity or small (PM $<\sim 1$), and the background field has significant interface normal component ($\sin\alpha \sim 1$). It is also required that $\beta_A = V_A/c \ll 1$ for the derivations to hold. This means that the Alfvén wave speed is much smaller than the light speed, such that the low-speed limit of the Lorentz transform is applicable.

4.3.1 Solution to horizontally-polarized system at insulating wall

Recall the boundary conditions and the associated modes in horizontally-polarized system, we write

$$\mathbf{f}^{\mathrm{AH-I}} + \mathbf{f}^{\mathrm{AH-R}} + \mathbf{f}^{\mathrm{BLH}} = \mathbf{f}_{z=0^-} = \begin{pmatrix} u_y \\ \tau_{zy} \\ b_y \\ e_x \end{pmatrix}_{z=0^-} = \begin{pmatrix} u_y \\ \tau_{zy} \\ b_y \\ e_x \end{pmatrix}_{z=0^+} = \mathbf{f}_{z=0^+} = \mathbf{f}^{\mathrm{SH}} + \mathbf{f}^{\mathrm{MH}}. \tag{120}$$

Plugging in all the expressions for the individual modes, we arrive at

$$\begin{pmatrix} 1 & 1 & 1 & -1 & 0 \\ -i\frac{\rho\omega\nu}{V_A}\frac{1-pV_A\cos\alpha}{\sin\alpha} & -i\frac{\rho\omega\nu}{V_A}\frac{-1-pV_A\cos\alpha}{\sin\alpha} & \rho V_A\sin\alpha\sqrt{\mathrm{Pm}} & \rho\beta\sqrt{1-p^2\beta^2} & 0 \\ -1 & 1 & -\sqrt{\mathrm{Pm}} & 0 & -1 \\ -V_A\sin\alpha & -V_A\sin\alpha & 0 & 0 & \frac{-i\eta k_z}{1+i\omega\eta/c^2} \end{pmatrix} \begin{pmatrix} u^0 \\ u^R \\ u^{\mathrm{BL}} \\ u^{\mathrm{SH}} \\ b^{\mathrm{MH}} \end{pmatrix} = \mathbf{0}.$$

Note that we are working in the scenario where the wall is an insulator. Therefore $\eta \to +\infty$, and $-i\eta k_z/(1 + i\omega\eta/c^2) \to -c^2 k_z/\omega$. In addition, in the insulator, the vertical wavenumber is given by $k_z = \frac{\omega}{c}\sqrt{1 - p^2c^2}$. Rearranging the terms, and setting incidence Alfvén wave $u^0 = 0$ as the reference amplitude, we have

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ i\frac{\rho\omega\nu}{V_A}\frac{1+pV_A\cos\alpha}{\sin\alpha} & \rho V_A \sin\alpha\sqrt{\mathrm{Pm}} & \rho\beta\sqrt{1-p^2\beta^2} & 0 \\ 1 & -\sqrt{\mathrm{Pm}} & 0 & -1 \\ -V_A\sin\alpha & 0 & 0 & -c\sqrt{1-p^2c^2} \end{pmatrix} \begin{pmatrix} u^R \\ u^{\mathrm{BL}} \\ u^{\mathrm{SH}} \\ b^{\mathrm{MH}} \end{pmatrix} = \begin{pmatrix} -1 \\ i\frac{\rho\omega\nu}{V_A}\frac{1-pV_A\cos\alpha}{\sin\alpha} \\ 1 \\ V_A\sin\alpha \end{pmatrix} \tag{121}$$

Nondimensionalizing the coefficients of the linear system gives

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ -i\epsilon_\eta \mathrm{Pm}\tilde{k}_- & \sqrt{\mathrm{Pm}}\sin\alpha & \gamma_{M_\beta}^{-1} & 0 \\ 1 & -\sqrt{\mathrm{Pm}} & 0 & -1 \\ -\sin\alpha & 0 & 0 & -\gamma_{\beta_A}^{-1} \end{pmatrix} \begin{pmatrix} u^R \\ u^{\mathrm{BL}} \\ u^{\mathrm{SH}} \\ b^{\mathrm{MH}} \end{pmatrix} = \begin{pmatrix} -1 \\ i\epsilon_\eta \mathrm{Pm}\tilde{k}_+ \\ 1 \\ \sin\alpha \end{pmatrix}. \tag{122}$$

The newly introduced dimensionless numbers are:

$$\begin{aligned} \tilde{k}_\pm &= \frac{\pm 1 - pV_A\cos\alpha}{\sin\alpha}, \\ \gamma_{M_\beta} &= \frac{V_A}{\beta}\frac{1}{\sqrt{1-p^2\beta^2}} = \frac{M_{A\beta}}{\sqrt{1-\bar{p}^2/M_{A\beta}^2}}, \\ \gamma_{\beta_A} &= \frac{V_A}{c}\frac{1}{\sqrt{1-p^2c^2}} = \frac{\beta_A}{\sqrt{1-\bar{p}^2/\beta_A^2}}, \end{aligned} \tag{123}$$

where $M_{A\beta} = V_A/\beta$ is the Alfvén Mach number with respect to S-wave, which gives the ratio of Alfvén wave velocity to shear wave speed; $\beta_A = V_A/c$ gives the ratio between Alfvén wave velocity and light speed. The reflection and transmission coefficients of the Alfvén wave at the boundary are then the solutions to the linear system, concisely written as

$$\mathbf{F}\mathbf{w} = \mathbf{f}^0. \tag{124}$$

This would be the form that every problem should eventually reduce to.

Under the conditions of the Earth's core, some dimensionless numbers are so extreme that simplifications can be safely made. Indeed, assuming an Alfvén wave speed of 0.05m/s in the fluid, and an S-wave speed of 7km/s in the solid, the Alfvén-S Mach number is $M_\beta \approx 10^{-5}$; the ratio of Alfvén wave speed to light speed is even smaller, given by $\beta_A \approx 10^{-10}$. Given that $\bar{p}$ is zero (normal incidence of travelling Alfvén wave) or of order unity (at some finite incidence angle, say between 5 to 85°), the dimensionless group $|\gamma_{M_\beta}| \sim M_{A_\beta} - M_{A_\beta}^2 \sim 10^{-5} - 10^{-10}$, respectively, and $|\gamma_{\beta_A}| \sim \beta_A - \beta_A^2 \sim 10^{-10} - 10^{-20}$. This means that the original linear equation can be reinterpreted in two ways. First, from the force balance indicated by the continuity of traction, u^{BL} has such dominant coefficient that, to leading order of $M_{A\beta}$, it indicates $u^{\mathrm{BL}} = 0$. The same thing can be said about b^{MH}, when analyzing the continuity of electric field to leading order of β_A. On the other hand, one can also argue that u^{BL} and b^{MH} must be at least $O(M_{A\beta})$ and $O(\beta_A)$ of the input field u^R, so much so that they must have negligibly small contributions in the continuity of velocity and magnetic field. Either way, the arguments lead to simplified system

$$\begin{pmatrix} 1 & 1 \\ 1 & -\sqrt{\mathrm{Pm}} \end{pmatrix} \begin{pmatrix} u^R \\ u^{\mathrm{BL}} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{125}$$

which gives exactly the same answer as in Schaeffer, Jault, et al. (2012)

$$u^R = \frac{1-\sqrt{\mathrm{Pm}}}{1+\sqrt{\mathrm{Pm}}}, \quad u^{\mathrm{BL}} = -\frac{2}{1+\sqrt{\mathrm{Pm}}}. \tag{126}$$

Without making such assumptions, however, we would then need to solve the whole 4-D system (eq.122). Making use of the symbolic mathematics package `SymPy`, we arrive at

$$
\begin{aligned}
u^R &= \frac{1-\sqrt{\mathrm{Pm}}+\gamma_{M_\beta}\left(\sqrt{\mathrm{Pm}}\sin\alpha+i\epsilon_\eta \mathrm{Pm}^{3/2}\tilde{k}_+\right)-\gamma_{\beta_A}\sin\alpha-\gamma_{M_\beta}\gamma_{\beta_A}\sqrt{\mathrm{Pm}}\sin^2\alpha}{1+\sqrt{\mathrm{Pm}}+\gamma_{M_\beta}\left(\sqrt{\mathrm{Pm}}\sin\alpha-i\epsilon_\eta \mathrm{Pm}^{3/2}\tilde{k}_-\right)+\gamma_{\beta_A}\sin\alpha+\gamma_{M_\beta}\gamma_{\beta_A}\sqrt{\mathrm{Pm}}\sin^2\alpha} \\
&= \frac{\left(1+\gamma_{M_\beta}\sqrt{\mathrm{Pm}}\sin\alpha\right)\left(1-\gamma_{\beta_A}\sin\alpha\right)-\sqrt{\mathrm{Pm}}+i\epsilon_\eta\gamma_{M_\beta}\mathrm{Pm}^{\frac{3}{2}}\tilde{k}_+}{\left(1+\gamma_{M_\beta}\sqrt{\mathrm{Pm}}\sin\alpha\right)\left(1+\gamma_{\beta_A}\sin\alpha\right)+\sqrt{\mathrm{Pm}}-i\epsilon_\eta\gamma_{M_\beta}\mathrm{Pm}^{\frac{3}{2}}\tilde{k}_-} \\
&= \frac{\left(1+M_\beta\sqrt{\mathrm{Pm}}\frac{\sin\alpha}{\sqrt{1-p^2\beta^2}}\right)\left(1-\beta_A\frac{\sin\alpha}{\sqrt{1-p^2c^2}}\right)-\sqrt{\mathrm{Pm}}+iM_\beta\epsilon_\eta\mathrm{Pm}^{\frac{3}{2}}\frac{\sin\alpha(1-\bar{p}\cos\alpha)}{\sqrt{1-p^2\beta^2}}}{\left(1+M_\beta\sqrt{\mathrm{Pm}}\frac{\sin\alpha}{\sqrt{1-p^2\beta^2}}\right)\left(1+\beta_A\frac{\sin\alpha}{\sqrt{1-p^2c^2}}\right)+\sqrt{\mathrm{Pm}}+iM_\beta\epsilon_\eta\mathrm{Pm}^{\frac{3}{2}}\frac{\sin\alpha(1+\bar{p}\cos\alpha)}{\sqrt{1-p^2\beta^2}}},
\end{aligned} \tag{127}
$$

and

$$
\begin{aligned}
u^{\mathrm{BL}} &= \frac{-2+i\epsilon_\eta\gamma_{M_\beta}\mathrm{Pm}\left((1+\gamma_{\beta_A}\sin\alpha)\tilde{k}_++(1-\gamma_{\beta_A}\sin\alpha)\tilde{k}_-\right)}{\left(1+\gamma_{M_\beta}\sqrt{\mathrm{Pm}}\sin\alpha\right)\left(1+\gamma_{\beta_A}\sin\alpha\right)+\sqrt{\mathrm{Pm}}-i\epsilon_\eta\gamma_{M_\beta}\mathrm{Pm}^{\frac{3}{2}}\tilde{k}_-} \\
&= \frac{-2\left(1+i\frac{M_\beta}{\sqrt{1-p^2\beta^2}}\epsilon_\eta\mathrm{Pm}\left(\bar{p}\cot\alpha-\frac{\beta_A}{\sqrt{1-p^2c^2}}\right)\right)}{\left(1+M_\beta\sqrt{\mathrm{Pm}}\frac{\sin\alpha}{\sqrt{1-p^2\beta^2}}\right)\left(1+\beta_A\frac{\sin\alpha}{\sqrt{1-p^2c^2}}\right)+\sqrt{\mathrm{Pm}}+iM_\beta\epsilon_\eta\mathrm{Pm}^{\frac{3}{2}}\frac{\sin\alpha(1+\bar{p}\cos\alpha)}{\sqrt{1-p^2\beta^2}}}.
\end{aligned} \tag{128}
$$

The coefficients for the elastic and the electromagnetic waves are given by

$$
\begin{aligned}
u^{\mathrm{SH}} &= \gamma_{M_\beta}\frac{2\sqrt{\mathrm{Pm}}\sin\alpha+i\mathrm{Pm}\epsilon_\eta\gamma_{\beta_A}\sin\alpha(\tilde{k}_+-\tilde{k}_-)+i\mathrm{Pm}\epsilon_\eta(\tilde{k}_++\tilde{k}_-)+i\mathrm{Pm}^{\frac{3}{2}}\epsilon_\eta(\tilde{k}_+-\tilde{k}_-)}{\left(1+\gamma_{M_\beta}\sqrt{\mathrm{Pm}}\sin\alpha\right)\left(1+\gamma_{\beta_A}\sin\alpha\right)+\sqrt{\mathrm{Pm}}-i\epsilon_\eta\gamma_{M_\beta}\mathrm{Pm}^{\frac{3}{2}}\tilde{k}_-} \\
&= \frac{2M_\beta\sqrt{\mathrm{Pm}}}{\sqrt{1-p^2\beta^2}}\frac{\sin\alpha+i\epsilon_\eta\mathrm{Pm}\csc\alpha-i\epsilon_\eta\sqrt{\mathrm{Pm}}\bar{p}\cot\alpha+i\frac{\beta_A}{\sqrt{1-p^2c^2}}\epsilon_\eta\sqrt{\mathrm{Pm}}}{\left(1+M_\beta\sqrt{\mathrm{Pm}}\frac{\sin\alpha}{\sqrt{1-p^2\beta^2}}\right)\left(1+\beta_A\frac{\sin\alpha}{\sqrt{1-p^2c^2}}\right)+\sqrt{\mathrm{Pm}}+iM_\beta\epsilon_\eta\mathrm{Pm}^{\frac{3}{2}}\frac{\sin\alpha(1+\bar{p}\cos\alpha)}{\sqrt{1-p^2\beta^2}}}
\end{aligned} \tag{129}
$$

$$
\begin{aligned}
u^{\mathrm{MH}} &= \gamma_{\beta_A}\sin\alpha\frac{-2\left(1+\gamma_{M_\beta}\sqrt{\mathrm{Pm}}\sin\alpha\right)-i\epsilon_\eta\gamma_{M_\beta}\mathrm{Pm}^{\frac{3}{2}}(\tilde{k}_+-\tilde{k}_-)}{\left(1+\gamma_{M_\beta}\sqrt{\mathrm{Pm}}\sin\alpha\right)\left(1+\gamma_{\beta_A}\sin\alpha\right)+\sqrt{\mathrm{Pm}}-i\epsilon_\eta\gamma_{M_\beta}\mathrm{Pm}^{\frac{3}{2}}\tilde{k}_-} \\
&= \frac{\beta_A^2}{\sqrt{1-p^2c^2}}\frac{-2\sin\alpha\left(1+M_\beta\sqrt{\mathrm{Pm}}\frac{\sin\alpha}{\sqrt{1-p^2\beta^2}}\right)-i2M_\beta\epsilon_\eta\mathrm{Pm}^{\frac{3}{2}}\frac{1}{\sqrt{1-p^2\beta^2}}}{\left(1+M_\beta\sqrt{\mathrm{Pm}}\frac{\sin\alpha}{\sqrt{1-p^2\beta^2}}\right)\left(1+\beta_A\frac{\sin\alpha}{\sqrt{1-p^2c^2}}\right)+\sqrt{\mathrm{Pm}}+iM_\beta\epsilon_\eta\mathrm{Pm}^{\frac{3}{2}}\frac{\sin\alpha(1+\bar{p}\cos\alpha)}{\sqrt{1-p^2\beta^2}}}
\end{aligned} \tag{130}
$$

These are just documented here for completeness, but our major interest is in the reflected Alfvén wave (eq.127) and perhaps the boundary layer behaviour (eq.128). Indeed, we see that the transmitted SH wave and the transmitted electromagnetic wave have prefactors γ_{M_β} and γ_{β_A}, respectively. At relatively small γ_{M_β} and γ_{β_A}, the two waves have amplitudes

$$
u^{\mathrm{SH}} \approx \gamma_{M_\beta}\frac{2\sqrt{\mathrm{Pm}}\sin\alpha}{1+\sqrt{\mathrm{Pm}}} \sim \gamma_{M_\beta}, \qquad u^{\mathrm{MH}} \approx -\gamma_{\beta_A}\frac{2\sin\alpha}{1+\sqrt{\mathrm{Pm}}} \sim \gamma_{\beta_A}
$$

that scale with γ_{M_β} and γ_{β_A}, respectively. Therefore, both waves have negligibly small amplitudes in comparison to the Alfvén waves and the Hartmann layer at the condition of the Earth's core (or virtually any MHD scenario where $V_A \ll \beta \ll c$), although they contibute non-trivially to the stress field and the electric field at the boundary. Meanwhile, the reflected Alfvén wave and the Hartmann layer solutions revert back to the simplified case (eq.126) at $\gamma_{M_\beta} \ll 1$ and $\gamma_{\beta_A} \ll 1$.

As a final remark, recall that throughout the derivation, we actually only used the so-called "zeroth-order" approximations of the vertical wavenumber for the Alfvén wave and Hartmann layer. Each element in the linear system (eq.122) that corresponds to the MHD waves comes with a relative error of $O(\epsilon_\eta)$. Taking that into account, the reflected wave coefficients are more properly written as follows

$$\begin{aligned} u^R &= \frac{\left(1+\gamma_{M_\beta}\sqrt{\mathrm{Pm}}\sin\alpha\right)(1-\gamma_{\beta_A}\sin\alpha) - \sqrt{\mathrm{Pm}} + O(\epsilon_\eta)}{\left(1+\gamma_{M_\beta}\sqrt{\mathrm{Pm}}\sin\alpha\right)(1+\gamma_{\beta_A}\sin\alpha) + \sqrt{\mathrm{Pm}} + O(\epsilon_\eta)}, \\ u^{\mathrm{BL}} &= \frac{-2+O(\epsilon_\eta)}{\left(1+\gamma_{M_\beta}\sqrt{\mathrm{Pm}}\sin\alpha\right)(1+\gamma_{\beta_A}\sin\alpha) + \sqrt{\mathrm{Pm}} + O(\epsilon_\eta)}. \end{aligned} \tag{131}$$

It follows that when $\gamma_{M_\beta} <\sim \epsilon_\eta$ and $\gamma_{\beta_A} <\sim \epsilon_\eta$, the additional terms sadly do not necessarily yield more accurate results than the simplified relation. Only when some of these dimensionless numbers are magnitude(s) larger than the inverse Lundquist number can these terms be actually used. In particular, one might expect the role of $\gamma_{M_\beta}\sqrt{\mathrm{Pm}}\sin\alpha$ to be important when the Alfvén wave speed is somewhat close to the longitudinal wave speed in the bordering elastic medium.

4.3.2 Solution to vertically-polarized system at insulating wall

4.4 Solutions at conductive wall

4.4.1 Solution to horizontally-polarized system at conductive wall

4.4.2 Solution to vertically-polarized system at conductive wall

4.5 Magneto-acoustic/elastic waves

We see that in some scenarios, the incompressible approximation needs to be relaxed to permit acoustic waves in the medium. Moreover, elastic waves are introduced in the bordering wall to account for the remanent displacement and stress. A natural question is, what are the electromagnetic effects of these waves? Conversely, one can ask what are the mechanical effects of electromagnetic waves? In this section, I shall look at the waves (modes) in the medium when both elasticity and electromagnetic properties are taken into account.

4.5.1 Magneto-acoustic waves

In fluid, we consider the following linearized set of governing equations,

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= -\nabla\frac{p}{\rho_0} + \frac{1}{\rho_0\mu_0}(\nabla\times\mathbf{b})\times\mathbf{B}_0 + \nu\nabla\cdot\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T - \frac{2}{3}\nabla\cdot\mathbf{u}\mathbf{I}\right), \\ \frac{\partial\rho}{\partial t} &+ \rho_0\nabla\cdot\mathbf{u} = 0, \\ \frac{p}{p_0} &= \gamma\frac{\rho}{\rho_0}, \\ \frac{\partial\mathbf{b}}{\partial t} &= \nabla\times(\mathbf{u}\times\mathbf{B}_0) + \eta\nabla^2\mathbf{b}. \end{aligned} \tag{132}$$

Here I assume the background velocity field $\mathbf{u}_0 = \mathbf{0}$, and a uniform background magnetic field $\mathbf{B}_0$. The background state is considered to be in hydrostatic equilibrium. All quantities without the 0 subscript, i.e. ρ, $\mathbf{u}$, p, $\mathbf{b}$, are all considered infinitesimal perturbations. The equation of state $p/p_0 = \gamma\rho/\rho_0$ takes a further assumption that the process is isentropic. Considering the potentially low frequency of the waves, this assumption may be problematic. Note that $\nabla \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^T - (2/3)\nabla \cdot \mathbf{u}\mathbf{I})$ can be written as $\nabla^2 \mathbf{u} + (1/3)\nabla(\nabla \cdot \mathbf{u})$, the Lorentz force $(\nabla \times \mathbf{b}) \times \mathbf{B}_0 = \mathbf{B}_0 \cdot \nabla \mathbf{b} - \nabla \mathbf{b} \cdot \mathbf{B}_0$, and the induction term $\nabla \times (\mathbf{u} \times \mathbf{B}_0) = \mathbf{B}_0 \cdot \nabla \mathbf{u} - \mathbf{B}_0(\nabla \cdot \mathbf{u})$. The governing equation can be rewritten as

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= -\nabla \left(\frac{p}{\rho_0} + \frac{1}{\rho\mu_0}\mathbf{B}_0 \cdot \mathbf{b} - \frac{\nu}{3}\nabla \cdot \mathbf{u} \right) + \frac{1}{\rho\mu_0}\mathbf{B}_0 \cdot \nabla \mathbf{b} + \nu \nabla^2 \mathbf{u}, \\ \frac{\partial p}{\partial t} &= -\gamma p_0 \nabla \cdot \mathbf{u}, \\ \frac{\partial \mathbf{b}}{\partial t} &= \mathbf{B}_0 \cdot \nabla \mathbf{u} - \mathbf{B}_0 \nabla \cdot \mathbf{u} + \eta \nabla^2 \mathbf{b}. \end{aligned} \tag{133}$$

There are multiple ways to proceed from here. If one takes $S_\eta \gg 1$ and neglects the viscous and magnetic diffusion, one can take a time derivative of the Navier-Stokes equation and merge the three equations into one. Then one can use the plane wave ansatz and solve the eigenvalue problem. Here I skip the first step and directly transform the equations into frequency-wavenumber domain, and write

$$\begin{aligned} &\left((i\omega + \nu k^2)\mathbf{I} + \frac{\nu}{3}\mathbf{k}\mathbf{k}\right) \cdot \mathbf{u} - i\frac{\mathbf{k}}{\rho_0}p + \frac{i}{\rho\mu_0}\left((\mathbf{B}_0 \cdot \mathbf{k})\mathbf{I} - \mathbf{k}\mathbf{B}_0\right) \cdot \mathbf{b} = \mathbf{0}, \\ &i\gamma p_0 \mathbf{k} \cdot \mathbf{u} - i\omega p = 0, \\ &i\left((\mathbf{B}_0 \cdot \mathbf{k})\mathbf{I} - \mathbf{B}_0\mathbf{k}\right) \cdot \mathbf{u} + (i\omega + \eta k^2)\mathbf{b} = \mathbf{0} \end{aligned} \tag{134}$$

which gives seven linear scalar equations, summarized in the matrix form

$$\begin{pmatrix} (i\omega + \nu k^2)\mathbf{I} + \frac{\nu}{3}\mathbf{k}\mathbf{k} & -i\frac{\mathbf{k}}{\rho_0} & \frac{i}{\rho\mu_0}\left((\mathbf{B}_0 \cdot \mathbf{k})\mathbf{I} - \mathbf{k}\mathbf{B}_0\right) \\ i\gamma p_0 \mathbf{k}^T & -i\omega & \mathbf{0} \\ i\left((\mathbf{B}_0 \cdot \mathbf{k})\mathbf{I} - \mathbf{B}_0\mathbf{k}\right) & \mathbf{0} & (i\omega + \eta k^2)\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \\ \mathbf{b} \end{pmatrix} = \mathbf{0} \tag{135}$$

Without loss of generality, I take $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, and $\mathbf{k} = k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}$. The wave vector and the background field then defines the Oxz plane. Writing out the equation in components,

$$\begin{pmatrix} i\omega + \nu\left(k^2 + \frac{k_x^2}{3}\right) & 0 & \frac{\nu}{3}k_xk_z & -i\frac{k_x}{\rho_0} & i\frac{B_0k_z}{\rho_0\mu_0} & 0 & -i\frac{B_0k_x}{\rho\mu_0} \\ 0 & i\omega + \nu k^2 & 0 & 0 & 0 & i\frac{B_0k_z}{\rho_0\mu_0} & 0 \\ \frac{\nu}{3}k_xk_z & 0 & i\omega + \nu\left(k^2 + \frac{k_z^2}{3}\right) & -i\frac{k_z}{\rho_0} & 0 & 0 & 0 \\ i\gamma p_0k_x & 0 & i\gamma p_0k_z & -i\omega & 0 & 0 & 0 \\ iB_0k_z & 0 & 0 & 0 & i\omega + \eta k^2 & 0 & 0 \\ 0 & iB_0k_z & 0 & 0 & 0 & i\omega + \eta k^2 & 0 \\ -iB_0k_x & 0 & 0 & 0 & 0 & 0 & i\omega + \eta k^2 \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \\ p \\ b_x \\ b_y \\ b_z \end{pmatrix} = \mathbf{0} \tag{136}$$

At this stage we apply the high Lundquist number approximation, and get rid of all diffusion terms. The simplified matrix equation is

$$\begin{pmatrix} i\omega & 0 & 0 & -i\frac{k_x}{\rho_0} & i\frac{B_0k_z}{\rho_0\mu_0} & 0 & -i\frac{B_0k_x}{\rho\mu_0} \\ 0 & i\omega & 0 & 0 & 0 & i\frac{B_0k_z}{\rho_0\mu_0} & 0 \\ 0 & 0 & i\omega & -i\frac{k_z}{\rho_0} & 0 & 0 & 0 \\ i\gamma p_0k_x & 0 & i\gamma p_0k_z & -i\omega & 0 & 0 & 0 \\ iB_0k_z & 0 & 0 & 0 & i\omega & 0 & 0 \\ 0 & iB_0k_z & 0 & 0 & 0 & i\omega & 0 \\ -iB_0k_x & 0 & 0 & 0 & 0 & 0 & i\omega \end{pmatrix} \begin{pmatrix} u_x \\ u_y \\ u_z \\ p \\ b_x \\ b_y \\ b_z \end{pmatrix} = \mathbf{0} \tag{137}$$

Despite its apparent complexity, one can exploit the sparsity of the matrix and the many common factors, and use elementary row operations to simplify the matrices. Note that these elementary row operations are just algebraic equivalence of subtitution of variables when working with differential equations. These row operations yield the following equivalent equation

$$\begin{pmatrix} -\omega^2+\alpha^2k_x^2+\frac{B_0^2}{\rho_0\mu_0}k^2 & 0 & \alpha^2k_xk_z & 0 & 0 & 0 & 0 \\ 0 & -\omega^2+\frac{B_0^2}{\rho_0\mu_0}k_z^2 & 0 & 0 & 0 & 0 & 0 \\ \alpha^2k_xk_z & 0 & -\omega^2+\alpha^2k_z^2 & 0 & 0 & 0 & 0 \\ \rho_0\alpha^2k_x & 0 & \rho_0\alpha^2k_z & -\omega & 0 & 0 & 0 \\ B_0k_z & 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & B_0k_z & 0 & 0 & 0 & \omega & 0 \\ -B_0k_z & 0 & 0 & 0 & 0 & 0 & \omega \end{pmatrix}\begin{pmatrix} u_x \\ u_y \\ u_z \\ p \\ b_x \\ b_y \\ b_z \end{pmatrix} = \mathbf{0}. \tag{138}$$

The almost-lower-triangular structure (or block-diagonal, depending on how this is viewed) makes the determinant of the matrix a piece of cake. The determinant is simply

$$\det\mathbf{A} = -\omega^4\begin{vmatrix} -\omega^2+\alpha^2k_x^2+V_A^2k^2 & 0 & \alpha^2k_xk_z \\ 0 & -\omega^2+V_A^2k_z^2 & 0 \\ \alpha^2k_xk_z & 0 & -\omega^2+\alpha^2k_z^2 \end{vmatrix} = 0 \tag{139}$$

The degree-10 polynomial has four trivial roots, $\omega_7 = \omega_8 = \omega_9 = \omega_{10} = 0$. It does not mean that the original system has four-fold degeneracy, however, since three of them are apparently artificially added during elementary row operations to avoid having ω on the denominator. Plugging $\omega = 0$ into the original matrix, we find that the zero-frequency mode somehow fulfills $b_y = 0$ and $k_zb_x - k_xb_z = 0$. However, due to the solenoidal condition, $k_xb_x + k_zb_z = 0$. Augmenting these two equations, we see that unless $k_x^2 + k_z^2 = k^2 = 0$, $\mathbf{b}$ has to be zero. Therefore, the only non-trivial mode that is permitted when $\omega = 0$ is a solution with uniform magnetic/velocity/pressure field. However, I postulate that, in the presence of magnetic dissipation, this solution might be the one that supports the Hartmann layer branch. The rest six roots are given by the following equation

$$\begin{aligned} \left(-\omega^2+V_A^2k_z^2\right)\left[\left(-\omega^2+\alpha^2k_x^2+V_A^2k^2\right)\left(-\omega^2+\alpha^2k_z^2\right)-\alpha^4k_x^2k_z^2\right] &= 0 \\ \left(\omega^2-V_A^2k_z^2\right)\left(\omega^4-\omega^2k^2\left(\alpha^2+V_A^2\right)+\alpha^2V_A^2k_z^2k^2\right) &= 0 \end{aligned} \tag{140}$$

which gives the following roots

$$\begin{aligned} \omega_{1,2}^2 &= V_A^2k_z^2 \\ \omega_{3,4}^2 &= \frac{k^2}{2}\left(\left(\alpha^2+V_A^2\right)+\sqrt{(\alpha^2+V_A^2)^2-4\alpha^2V_A^2\frac{k_z^2}{k^2}}\right), \\ \omega_{5,6}^2 &= \frac{k^2}{2}\left(\left(\alpha^2+V_A^2\right)-\sqrt{(\alpha^2+V_A^2)^2-4\alpha^2V_A^2\frac{k_z^2}{k^2}}\right). \end{aligned} \tag{141}$$

Among these roots, $\omega_{1,2}$ give the exact dispersion relation for the Alfvén wave; these thus give the Alfvén waves as we know. The two roots correspond to the waves travelling downwind and upwind the background field. The eigenmodes that correspond to these eigenvalues are given by $(u_y, b_y) = (1, -B_0k_z/\omega) = (1, -\sqrt{\rho_0\mu_0}V_Ak_z/\omega) = (1, \mp\sqrt{\rho\mu_0})$, while the rest of the components are set to zero. Therefore, the waves that correspond to the exact Alfvén wave dispersion relations comprise of velocity and magnetic fields only polarized in the y direction, perpendicular to both the wave vector $\mathbf{k}$ and the background field $\mathbf{B}_0$. These are previously referred to as the horizontally polarized Alfvén wave (AH).

For the other four roots, known as the fast ($\omega_{3,4}$) and the slow ($\omega_{5,6}$) magneto-acoustic waves, I try to expand the expression in small Alfvén Mach number limit. The Alfvén Mach number is defined as

$M_A = V_A/\alpha$. At $M_A \ll 1$, we have the fast magneto-acoustic wave

$$
\begin{aligned}
\omega_{3,4}^2 &= \frac{k^2}{2}\left(\alpha^2\left(1+M_A^2\right)+\alpha^2\sqrt{1+\left(2-4\frac{k_z^2}{k^2}\right)M_A^2+M_A^4}\right)\\
&\approx \frac{k^2}{2}\alpha^2\left(2+\left(2-2\frac{k_z^2}{k^2}\right)M_A^2+\frac{1}{2}M_A^4+O(M_A^4)\right)\\
&= \alpha^2k^2\left(1+\left(1-\frac{k_z^2}{k^2}\right)M_A^2+O\left(M_A^4\right)\right),
\end{aligned}
\tag{142}
$$

and the slow magneto-acoustic wave

$$
\begin{aligned}
\omega_{5,6}^2 &= \frac{k^2}{2}\left(\alpha^2\left(1+M_A^2\right)-\alpha^2\sqrt{1+\left(2-4\frac{k_z^2}{k^2}\right)M_A^2+M_A^4}\right)\\
&\approx \frac{k^2}{2}\alpha^2\left(2\frac{k_z^2}{k^2}M_A^2-\frac{1}{2}M_A^4+O(M_A^4)\right)\\
&= V_A^2k_z^2\left(1+O\left(M_A^2\right)\right).
\end{aligned}
\tag{143}
$$

It is hence shown that to the first order in M_A, the fast and slow magneto-acoustic waves share the same dispersion relation with pure acoustic and pure Alfvén waves, respectively. Note that an implicit assumption made here: k_z/k is bounded. Hereinafter I shall assume that k_z/k is around unity, in other words k_x not so high. Moreover, the eigenmodes correspond to these magneto-acoustic modes can be encoded as

$$
\mathbf{f} = \begin{pmatrix} u_x\\ u_y\\ u_z\\ p\\ b_x\\ b_y\\ b_z \end{pmatrix} = \begin{pmatrix} -\frac{\omega}{V_Ak_x}\\ 0\\ \frac{V_A^2k^2+\alpha^2k_x^2}{V_A\alpha^2k_x^2k_z}\omega-\frac{\omega^3}{V_A\alpha^2k_x^2k_z}\\ -\rho_0V_A\frac{k^2}{k_x^2}+\frac{\rho_0}{V_Ak_x^2}\omega^2\\ -\frac{k_z}{k_x}\sqrt{\rho_0\mu_0}\\ 0\\ \sqrt{\rho_0\mu_0} \end{pmatrix} = \frac{1}{k_x}\begin{pmatrix} -\frac{\omega}{V_A}\\ 0\\ \frac{\omega}{V_A}\frac{V_A^2k^2+\alpha^2k_x^2-\omega^2}{\alpha^2k_xk_z}\\ \frac{\rho_0V_A}{k_x}\left(-k^2+\frac{\omega^2}{V_A^2}\right)\\ -\sqrt{\rho_0\mu_0}k_z\\ 0\\ \sqrt{\rho_0\mu_0}k_x \end{pmatrix}.
\tag{144}
$$

For the fast magneto-acoustic mode, we plug in $\omega_{3,4} = \pm\alpha k \pm M_AV_Ak(1-k_z^2/k_x^2)/2+O\left(M_A^4\right)$ to simplify the expression

$$
\mathbf{f}_{3,4} = \frac{1}{k_x}\begin{pmatrix} \mp\frac{\alpha}{V_A}k+O\left(M_A^2\right)\\ 0\\ \mp\frac{k_z}{k_x}\frac{\alpha}{V_A}k+O\left(M_A^2\right)\\ \rho_0\frac{\alpha^2k^2}{V_Ak_x}+O\left(M_A^2\right)\\ -\sqrt{\rho_0\mu_0}k_z\\ 0\\ \sqrt{\rho_0\mu_0}k_x \end{pmatrix} \approx \frac{k^2}{k_x^2}\begin{pmatrix} \mp M_A^{-1}\frac{k_x}{k}\\ 0\\ \mp M_A^{-1}\frac{k_z}{k}\\ M_A^{-1}\rho_0\alpha\\ -\sqrt{\rho_0\mu_0}\frac{k_zk_x}{k^2}\\ 0\\ \sqrt{\rho_0\mu_0}\frac{k_x^2}{k^2} \end{pmatrix}.
\tag{145}
$$

Compared to Alfvén waves, where the amplitude is equal between the velocity field and the magnetic field, when $M_A \ll 1$, the amplitude of the fast magneto-acoustic waves are predominantly in the velocity field. In fact, when we take $M_A \ll 1$ and keep up to the leading order of M_A (M_A^{-1}) in each of the components, we recover the purely acoustic wave, where magnetic effects are negligible:

$$
\mathbf{f}_{3,4} = \begin{pmatrix} u_x\\ u_z\\ p \end{pmatrix} \approx \frac{k^2}{k_x^2}M_A^{-1}\begin{pmatrix} \mp\frac{k_x}{k}\\ \mp\frac{k_z}{k}\\ \rho_0\alpha \end{pmatrix}.
\tag{146}
$$

Note that this is simply an eigenvector, and all leading prefactors can be discarded. We thus conclude at low Alfvén Mach number limit, the fast magneto-acoustic modes simply degenerate into the ordinary acoustic waves in non-electrically-conducting fluid.

4.5.2 Magneto-elastic waves

Similar to magneto-acoustic waves, elastic medium which supports both compressional and longitudinal waves may also support waves coupling electromagnetic fields. These are termed magneto-elastic waves, and has been studied in the 1950s to 1960s, interestingly, following the development of magnetohydrodynamics. Much of this section has already been documented in e.g. Baños (1956), although the literature omits many intermediate steps.

First, I state that for perfect insulator, there will be no interaction between electromagnetic fields and velocity fields. Indeed, absence of electric currents removes the Lorentz force from the picture, prohibiting any electromagnetic feedback to the mechanical waves. On the other hand, there will be no induction in the insulator due to particle motion. It follows that all of the following derivations are based on some finite $\sigma > 0$, $\eta < +\infty$, such that the displacement currents are neglected.

The governing equation for the elastic medium in uniform background field reads

$$\begin{aligned} \rho\frac{\partial^2 \mathbf{U}}{\partial t^2} &= \nabla\cdot\left(\boldsymbol{\tau} + \boldsymbol{\tau}^M\right) = \nabla\cdot\left(\lambda(\nabla\cdot\mathbf{U})\mathbf{I} + 2\mu\left(\nabla\mathbf{U} + \nabla\mathbf{U}^T\right) + \frac{1}{\mu_0}\mathbf{B}\mathbf{B} - \frac{B^2}{2\mu_0}\mathbf{I}\right), \\ \frac{\partial \mathbf{B}}{\partial t} &= \nabla\times(\mathbf{u}\times\mathbf{B}) + \eta\nabla^2\mathbf{B} = \nabla\times\left(\frac{\partial\mathbf{U}}{\partial t}\right) + \eta\nabla^2\mathbf{B}, \end{aligned} \tag{147}$$

where $\boldsymbol{\tau}^M$ is the Maxwell stress. We already applied the infinitesimal deformation approximation on the velocity field. However, the Maxwell stress and the induction term remain to be linearized. Again, decomposing the magnetic field $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, and taking the time derivative of the the momentum equation, we arrive at the fully linearized equations of velocity and magnetic field

$$\begin{aligned} \frac{\partial^2\mathbf{u}}{\partial t^2} &= \frac{\lambda}{\rho_0}\nabla(\nabla\cdot\mathbf{u}) + \frac{\mu}{\rho_0}\left(\nabla^2\mathbf{u} + \nabla(\nabla\cdot\mathbf{u})\right) + \mathbf{V}_A\cdot\nabla\frac{\partial\mathbf{b}}{\partial t} - \nabla\frac{\partial\mathbf{b}}{\partial t}\cdot\mathbf{V}_A, \\ &= \left(\alpha^2 - \beta^2\right)\nabla(\nabla\cdot\mathbf{u}) + \beta^2\nabla^2\mathbf{u} + \mathbf{V}_A\cdot\nabla\frac{\partial\mathbf{b}}{\partial t} - \nabla\frac{\partial\mathbf{b}}{\partial t}\cdot\mathbf{V}_A \\ \frac{\partial\mathbf{b}}{\partial t} &= \mathbf{V}_A\cdot\nabla\mathbf{u} - \mathbf{V}_A\nabla\cdot\mathbf{u} + \eta\nabla^2\mathbf{b}. \end{aligned} \tag{148}$$

Note the same transformation $\mathbf{b} := \mathbf{b}/\sqrt{\rho\mu_0}$ has already been applied. The magnetic field is oriented in the z direction, i.e. $\mathbf{V}_A = V_A\hat{\mathbf{z}}$. Baños (1956) merged the entire Maxwell equations and momentum equation into one vector equation, which can then be converted to three scalar equations by taking the z-component, the divergence, and the z-component of the curl. However, this approach requires tedious and lengthy derivation beforehand to first come to the single vector equation, as well as yields high order systems. I follow the same procedure as the magneto-acoustic wave, and formulate the equation as a 6×6 matrix. Note here the stress tensor has been explicit written out using the constitutive relation, which removes the need to include the constitutive relation (counterpart of $\partial_t p = -\gamma p_0 \nabla\cdot\mathbf{u}$ in magneto-acoustic wave derivations) in the system. Using the plane wave ansatz,

$$\begin{aligned} & -\omega^2\mathbf{u} + \left(\alpha^2 - \beta^2\right)\mathbf{k}\left(\mathbf{k}\cdot\mathbf{u}\right) + \beta^2 k^2\mathbf{u} + \omega\mathbf{k}\left(\mathbf{V}_A\cdot\mathbf{b}\right) - \omega\left(\mathbf{V}_A\cdot\mathbf{k}\right)\mathbf{b} = \mathbf{0} \\ \Longleftrightarrow\quad & \left[\left(-\omega^2 + \beta^2 k^2\right)\mathbf{I} + \left(\alpha^2 - \beta^2\right)\mathbf{k}\mathbf{k}\right]\cdot\mathbf{u} + \omega\left[\mathbf{k}\mathbf{V}_A - (\mathbf{k}\cdot\mathbf{V}_A)\mathbf{I}\right]\cdot\mathbf{b} = \mathbf{0} \\ & \left(i\omega + \eta k^2\right)\mathbf{b} + i(\mathbf{V}_A\cdot\mathbf{k})\mathbf{u} - i\mathbf{V}_A(\mathbf{k}\cdot\mathbf{u}) = \mathbf{0} \\ \Longleftrightarrow\quad & i\left[(\mathbf{V}_A\cdot\mathbf{k})\mathbf{I} - \mathbf{V}_A\mathbf{k}\right]\cdot\mathbf{u} + \left(i\omega + \eta k^2\right)\mathbf{I}\cdot\mathbf{b} = \mathbf{0} \end{aligned} \tag{149}$$

which forms the matrix equation

$$\begin{pmatrix} \left(-\omega^2+\beta^2k^2\right)\mathbf{I}+\left(\alpha^2-\beta^2\right)\mathbf{kk} & \omega\left[\mathbf{kV}_A-(\mathbf{k}\cdot\mathbf{V}_A)\mathbf{I}\right] \\ (\mathbf{V}_A\cdot\mathbf{k})\mathbf{I}-\mathbf{V}_A\mathbf{k} & \left(\omega-i\eta k^2\right)\mathbf{I} \end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \mathbf{b}\end{pmatrix}=\mathbf{0}.$$

This is still an eigenvalue problem, although since the system is not shaped as vector ODE, the corresponding matrix is not equivalent to a regular matrix eigenvalue problem, but contains higher powers of ω on the diagonal, also ω in off-diagonal elements:

$$\begin{pmatrix} -\omega^2+\alpha^2k_x^2+\beta^2k_z^2 & 0 & (\alpha^2-\beta^2)k_xk_z & -\omega k_zV_A & 0 & \omega k_xV_A \\ 0 & -\omega^2+\beta^2k^2 & 0 & 0 & -\omega k_zV_A & 0 \\ (\alpha^2-\beta^2)k_xk_z & 0 & -\omega^2+\alpha^2k_z^2+\beta^2k_x^2 & 0 & 0 & 0 \\ k_zV_A & 0 & 0 & \omega-i\eta k^2 & 0 & 0 \\ 0 & k_zV_A & 0 & 0 & \omega-i\eta k^2 & 0 \\ -k_xV_A & 0 & 0 & 0 & 0 & \omega-i\eta k^2 \end{pmatrix}\begin{pmatrix}u_x\\u_y\\u_z\\b_x\\b_y\\b_z\end{pmatrix}=\mathbf{0}. \tag{150}$$

If defining $M_\alpha=V_A/\alpha$ and $M_\beta=V_A/\beta$ as the Alfvén-P and Alfvén-S Mach numbers, respectively, the matrix can be rewritten as

$$\begin{pmatrix} M_\alpha^{-2}k_x^2+M_\beta^{-2}k_z^2-\frac{\omega^2}{V_A^2} & 0 & (M_\alpha^{-2}-M_\beta^{-2})k_xk_z & -\frac{\omega}{V_A}k_z & 0 & \frac{\omega}{V_A}k_x \\ 0 & M_\beta^{-2}k^2-\frac{\omega^2}{V_A^2} & 0 & 0 & -\frac{\omega}{V_A}k_z & 0 \\ (M_\alpha^{-2}-M_\beta^{-2})k_xk_z & 0 & M_\alpha^{-2}k_z^2+M_\beta^{-2}k_x^2-\frac{\omega^2}{V_A^2} & 0 & 0 & 0 \\ k_zV_A & 0 & 0 & \omega-i\eta k^2 & 0 & 0 \\ 0 & k_zV_A & 0 & 0 & \omega-i\eta k^2 & 0 \\ -k_xV_A & 0 & 0 & 0 & 0 & \omega-i\eta k^2 \end{pmatrix} \tag{151}$$

Solving the temporal branch (i.e. ω) is equivalent to solving a degree-9 polynomial. Fortunately, the matrix is readily separated into two blocks: the variables u_y and b_y are naturally decoupled from the rest of the system. This is similarly observed in magneto-acoustic waves, although there I didn't exploit this property. This simply splits the degree-9 polynomial into a cubic polynomial factor and a degree-6 polynomial factor. The cubic polynomial is given by

$$\begin{gathered} \det\mathbf{A}_1=\begin{vmatrix} -\omega^2+\beta^2k^2 & -k_z\omega V_A \\ k_zV_A & \omega-i\eta k^2\end{vmatrix}=-\left(\omega^2-\beta^2k^2\right)\left(\omega-i\eta k^2\right)+k_z^2V_A^2\omega=0 \\ \omega^3-i\eta k^2\omega^2-\left(\beta^2k^2+k_z^2V_A^2\right)\omega+i\eta\beta^2k^4=0 \end{gathered} \tag{152}$$

Using the Alfvén wavenumber as the characteristic length scale, the equation can be nondimensionalized

$$\left(\frac{\omega}{kV_A}\right)^3-i\epsilon_\eta\left(\frac{\omega}{kV_A}\right)^2-\left(M_\beta^{-2}+\frac{k_z^2}{k^2}\right)\left(\frac{\omega}{kV_A}\right)+i\epsilon_\eta M_\beta^{-2}=0 \tag{153}$$

where the wavenumber-dependent inverse Lundquist number is defined here as $\epsilon_\eta=\frac{\eta k}{V_A}$. At very low Alfvén-S Mach number, $M_\beta^{-2}\gg k_z^2/k^2$, the equation simplifies to a fully factorized form

$$\left[\left(\frac{\omega}{kV_A}\right)^2-M_\beta^{-2}\right]\left(\frac{\omega}{kV_A}-i\epsilon_\eta\right)=0. \tag{154}$$

Among the three roots of this polynomial,

$$\omega_{1,2}^2=k^2V_A^2M_\beta^{-2}=k^2\beta^2 \tag{155}$$

correspond to pure mechanical shear wave at low Mach number limit, and

$$\omega_3-i\eta k^2=0 \tag{156}$$

corresponds to the decaying solution, with skin depth $\sqrt{2\eta/\omega} = \sqrt{2/\mu_0\omega\sigma}$. The two scenarios should also be mainly mechanical (i.e. eigenvector has $|b|/|u| \sim O(M_\beta)$) and electromagnetic ($|u|/|b| \sim O(M_\beta)$), respectively. [How to show this? Brutal force solution, while still possible for this matrix, will soon be out of hand; how can I justify the accuracy of the solutions when making approximations on the coefficients? Is there an algebraic "condition number"?]

The second factor polynomial is given by the rather dense matrix

$$\det \mathbf{A}_2 = \det \begin{pmatrix} \alpha^2 k_x^2 + \beta^2 k_z^2 - \omega^2 & (\alpha^2 - \beta^2) k_x k_z & -\omega k_z V_A & \omega k_x V_A \\ (\alpha^2 - \beta^2) k_x k_z & \alpha^2 k_z^2 + \beta^2 k_x^2 - \omega^2 & 0 & 0 \\ k_z V_A & 0 & \omega - i\eta k^2 & 0 \\ -k_x V_A & 0 & 0 & \omega - i\eta k^2 \end{pmatrix}. \tag{157}$$

A few comments can be made on the submatrix above. First, the z-component of the momentum equation (second row) does not involve magnetic field. This can be easily justified in that there is no Lorentz force aligned with the background magnetic field in the linearized form. Second, similarly, the z-velocity, since it is aligned with the background field, does not contribute to the induction term (see third and fourth row). Expanding the determinant (hint: expand by the last column, and then expand by the last columns of the resulting submatrices),

$$\begin{aligned} \det \mathbf{A} &= (\omega k_x V_A)(\omega - i\eta k^2)(k_x V_A)\left(\alpha^2 k_z^2 + \beta^2 k_x^2 - \omega^2\right) \\ &\quad + (\omega - i\eta k^2)^2 (\omega^2 - \alpha^2 k^2)(\omega^2 - \beta^2 k^2) \\ &\quad + (\omega - i\eta k^2)(\omega k_z V_A)(k_z V_A)\left(\alpha^2 k_z^2 + \beta^2 k_x^2 - \omega^2\right) \\ &= \left(\omega - i\eta k^2\right)\left[\omega k^2 V_A^2 \left(\alpha^2 k_z^2 + \beta^2 k_x^2 - \omega^2\right) + \left(\omega - i\eta k^2\right)\left(\omega^2 - \alpha^2 k^2\right)\left(\omega^2 - \beta^2 k^2\right)\right] \\ &= \left(\omega - i\eta k^2\right)\left[\omega^5 - i\eta k^2 \omega^4 - \left(k^2 V_A^2 + (\alpha^2 + \beta^2) k^2\right)\omega^3\right. \\ &\qquad \left. + i\eta k^4 (\alpha^2 + \beta^2)\omega^2 + \left(\alpha^2 \beta^2 k^4 + \left(\alpha^2 k_z^2 + \beta^2 k_x^2\right) k^2 V_A^2\right)\omega - i\eta \alpha^2 \beta^2 k^6\right] \\ &= \left(\omega - i\eta k^2\right)\left[\omega^4 \left(\omega - i\eta k^2\right) - \left(\alpha^2 + \beta^2 + V_A^2\right) k^2 \omega^3\right. \\ &\qquad \left. + i\eta k^4 \left(\alpha^2 + \beta^2\right)\omega^2 + \left(\alpha^2 \beta^2 + \alpha^2 V_A^2 \frac{k_z^2}{k^2} + \beta^2 V_A^2 \frac{k_x^2}{k^2}\right) k^4 \omega - i\eta \alpha^2 \beta^2 k^6\right]. \end{aligned} \tag{158}$$

The roots of the degree-6 polynomial give the temporal branch of the magneto-elastic normal modes that couple the fields in the plane formed by $\mathbf{k}$ and $\mathbf{B}_0$. To our delight, we find that at least one solution can be exactly extracted:

$$\omega = i\eta k^2$$

which seems to correspond to the decaying magnetic field. However, substituting the relation back into the matrix, we find the eigenvector (normal mode) is indicated by $u_x = u_z = 0$, $-k_z b_x + k_x b_z = 0$. Augmenting the latter with solenoidal condition $k_x b_x + k_z b_z = 0$, we again conclude the only condition for nontrivial solution is $k_x^2 + k_z^2 = k^2 = 0$. This in turn implies $\omega = 0$.

The rest of the roots are unfortunately buried in the un-factorizable quintic equation

$$\begin{aligned} \omega^5 - i\eta k^2 \omega^4 - \left(k^2 V_A^2 + (\alpha^2 + \beta^2) k^2\right)\omega^3 + i\eta k^4 (\alpha^2 + \beta^2)\omega^2 \\ + \left(\alpha^2 \beta^2 k^4 + \left(\alpha^2 k_z^2 + \beta^2 k_x^2\right) k^2 V_A^2\right)\omega - i\eta \alpha^2 \beta^2 k^6 = 0 \end{aligned}$$

where a universal radical expression of roots is absent. Nevertheless, I shall discuss two approaches to simplify the problem. If non-dimensionalizing the equation using compressional wave wave length β/α

as the characteristic wavelength $1/k$, it can be rewritten as

$$
\begin{aligned}
&\left(\frac{\omega}{k\alpha}\right)^5 - i\frac{\eta k}{\alpha}\left(\frac{\omega}{k\alpha}\right)^4 - \left(1 + \frac{V_A^2}{\alpha^2} + \frac{\beta^2}{\alpha^2}\right)\left(\frac{\omega}{k\alpha}\right)^3 + i\frac{\eta k}{\alpha}\left(1 + \frac{\beta^2}{\alpha^2}\right)\left(\frac{\omega}{k\alpha}\right)^2 \\
&\quad + \left(\frac{\beta^2}{\alpha^2} + \frac{V_A^2}{\alpha^2}\left(\frac{k_z^2}{k^2} + \frac{\beta^2}{\alpha^2}\frac{k_x^2}{k^2}\right)\right)\frac{\omega}{k\alpha} - i\frac{\eta k}{\alpha}\frac{\beta^2}{\alpha^2} = 0, \\
&\left(\frac{\omega}{k\alpha}\right)^5 - i\epsilon_\eta M_\alpha\left(\frac{\omega}{k\alpha}\right)^4 - \left(1 + M_\alpha^2 + \Gamma^{-2}\right)\left(\frac{\omega}{k\alpha}\right)^3 + i\epsilon_\eta M_\alpha\left(1 + \Gamma^{-2}\right)\left(\frac{\omega}{k\alpha}\right)^2 \\
&\quad + \left(\Gamma^{-2} + M_\alpha^2\left(\frac{k_z^2}{k^2} + \Gamma^{-2}\frac{k_x^2}{k^2}\right)\right)\frac{\omega}{k\alpha} - i\epsilon_\eta M_\alpha \Gamma^{-2} = 0,
\end{aligned}
\tag{159}
$$

where I introduced the P- to S-wave velocity ratio $\Gamma = \alpha/\beta$. For a Poisson solid ($\lambda = \mu$), this ratio is $\Gamma = \sqrt{3} \approx 1.732 \sim 1$. Most typical solids, including virtually all of Earth's mantle, are at the same order of $\Gamma \sim 1$. We are ready to introduce simplifications. Just as in magneto-acoustic waves, the system with all three ingredients - background field, magnetic diffusion and elasticity - is not analytically tractable. However, removing any one of these would yield the system tractable. At very low Alfvén-P Mach number, the M_α^2 terms in the cubic and linear coefficients can be neglected compared to the other order unity term. In this case, the equation simplifies into a form where the background magnetic field is subordinate:

$$
\begin{aligned}
&\left(\frac{\omega}{k\alpha}\right)^5 - i\epsilon_\eta M_\alpha\left(\frac{\omega}{k\alpha}\right)^4 - \left(1 + \Gamma^{-2}\right)\left(\frac{\omega}{k\alpha}\right)^3 + i\epsilon_\eta M_\alpha\left(1 + \Gamma^{-2}\right)\left(\frac{\omega}{k\alpha}\right)^2 + \Gamma^{-2}\frac{\omega}{k\alpha} - i\epsilon_\eta M_\alpha \Gamma^{-2} = 0, \\
&\omega^5 - i\eta k^2\omega^4 - \left(\alpha^2 + \beta^2\right)k^2\omega^3 + i\eta k^4\left(\alpha^2 + \beta^2\right)\omega^2 + \alpha^2\beta^2k^4\omega - i\eta\alpha^2\beta^2k^6 = 0, \\
&\left(\omega - i\eta k^2\right)\left(\omega^2 - \alpha^2k^2\right)\left(\omega^2 - \beta^2k^2\right) = 0
\end{aligned}
\tag{160}
$$

reverting back to the scenario with no background field. The three solutions are: pure decaying (temporal branch behaviour) / evanescent (spatial branch behaviour) magnetic field governed by $\omega = i\eta k^2$, pure elastic compressional wave $\omega = \pm\alpha k$, and pure elastic shear wave $\omega = \pm\beta k$. The latter two pure elastic waves are also given as in the "weak-field" limit discussed in Baños (1956).

A Reflection and transmission at fluid interface

The behaviour of Alfvén waves at fluid-fluid intrfaces is more complicated since the Alfvén wave solution needs to be constructed simultaneously in both sides of the interface. This problem has, however, been studied by Ferraro (1954) in the dawn of MHD, albeit for ideal (diffusionless) case only. Although his derivations and results are self-consistent and seem reasonable, it seems peculiar to use electromagnetic continuity condition when the media are assumed to be ideal/diffusionless/infinitely conductive. Nevertheless, this provides a very comprehensive step as the orientations are fully taken into account, and is hence documented here.

A.1 Interface of two ideal fluids

This scenario is treated in Ferraro (1954). The media on both sides of the interface are considered ideal, i.e. free of both viscous and magnetic diffusion.

A.1.1 Problem setup

We consider the incidence of Alfvén wave at an oblique angle with the fluid-fluid interface in uniform $\mathbf{B}_0$. The two media have different densities, denoted as ρ_1 and ρ_2, which creates the discontinuity. The setup of geometry and coordinate systems is given in the box that follows.

Orientations in the system

We first recall the treatment of seismic wave reflection and transmission in isotropic medium. There are three orientations involved in the system:

1. orientation of the interface (described by plane normal $\hat{\mathbf{n}}$),
2. the wave vector of the wave $\mathbf{k}$, and
3. the polarization of the wave $\mathbf{A}_0$.

After defining the plane of incidence using $\hat{\mathbf{n}}$ and $\mathbf{k}$ and splitting the wavefield into P, SV and SH polarizations, the picture is fixed and the problem is relatively simple to solve. The only variables are the incidence, reflection and refraction angles, whose relations can be derived solely in terms of medium properties. This is however not the case for Alfvén waves. Due to the anisotropy introduced by the $\mathbf{B}_0$, there is a fourth distinctive orientation in the system

4. orientation of the background magnetic field $\mathbf{B}_0$.

This is not at all a trivial orientation, as eq.(28) shows this is the direction in which energy and information propagates. One can already anticipate that medium properties will not be the only thing present in the laws of reflection and refraction. Furthermore, since we have one more direction in the system, it is now ambiguous how to define the coordinate system. In this article I follow Ferraro's treatment and define three coordinate systems as follows

- Axis $\hat{\mathbf{z}}$ is defined to be the normal of the plane, forming an acute angle with the background field, i.e. $\hat{\mathbf{z}} = \hat{\mathbf{n}}$;
- Axis $\hat{\mathbf{x}}$ is so defined that $\mathbf{B}_0$ is in the plane Oxz and forms an acute angle with $\hat{\mathbf{x}}$; $\hat{\mathbf{y}}$ follows so that $Oxyz$ forms the right-hand system;
- Axis $\hat{\mathbf{z}}'$ is defined as $\widehat{\mathbf{B}}_0$; $Ox'yz'$ system is obtained by rotating $Oxyz$ around $\hat{\mathbf{y}}$;
- Axis $\hat{\mathbf{x}}''$ is so defined that $\mathbf{k}$ is in the plane $Ox''z$ and forms an acute angle with $\hat{\mathbf{x}}''$; $\hat{\mathbf{y}}''$ follows so that $Ox''y''z$ forms the right-hand system.

The three coordinate systems are

1. $Oxyz$, where $\hat{\mathbf{z}} = \hat{\mathbf{n}}$, and $\mathbf{B}_0$ defines Oxz;
2. $Ox'yz'$, where $\hat{\mathbf{z}}' = \widehat{\mathbf{B}}_0$, and the system shares the same $\hat{\mathbf{y}}$ with $Oxyz$;
3. $Ox''y''z$, where $\hat{\mathbf{z}} = \hat{\mathbf{n}}$, and $\mathbf{k}$ defines $Ox''z''$.

Unless otherwise specified, the three-tuple (a, b, c) represents three components in $Oxyz$ coordinates. Introducing angle $\beta =< \mathbf{B}_0, \hat{\mathbf{z}} >$ so that $\widehat{\mathbf{B}}_0 = (\sin\beta, 0, \cos\beta)$, angles θ, ϕ so that $\mathbf{k} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, the three coordinate systems are related via

$$\mathbf{r}' = \begin{pmatrix} x' \\ y \\ z' \end{pmatrix} = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{Q}'\mathbf{r}, \qquad \mathbf{r}'' = \begin{pmatrix} x'' \\ y'' \\ z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{Q}''\mathbf{r} \tag{161}$$

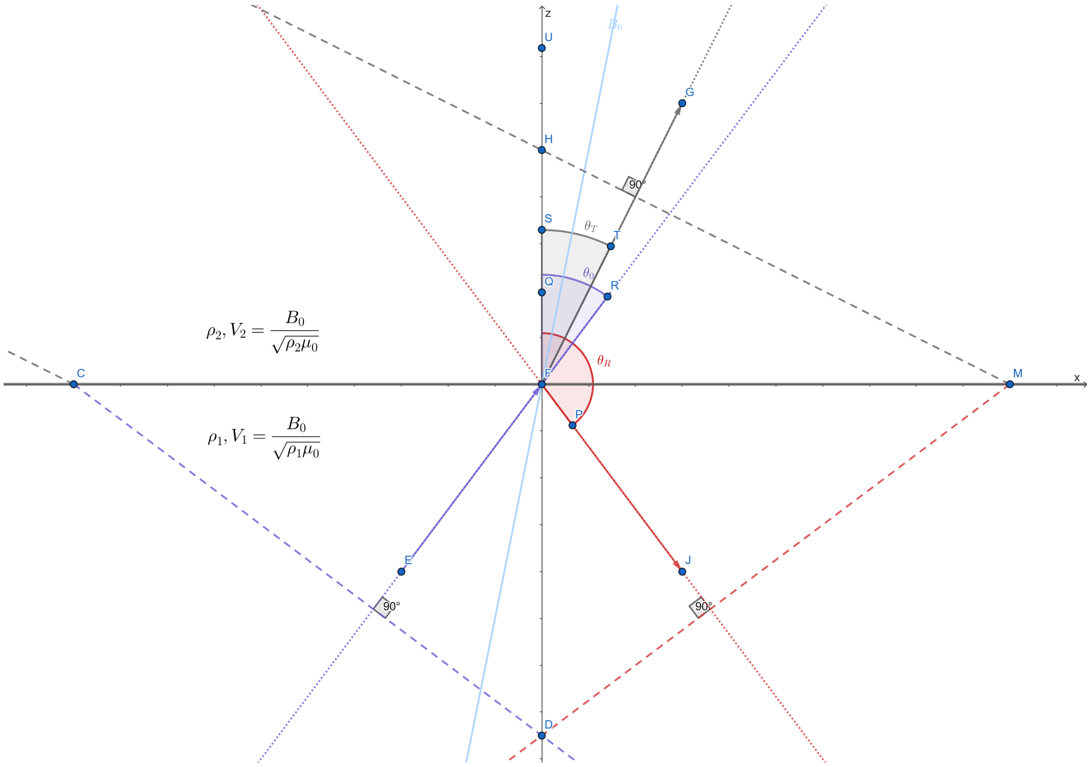

Figure 8: Setup of the Alfvén reflection problem at the fluid-fluid interface.

Boundary conditions (BCs) at the interface are the link between the solutions within two domains separated by the interface. For the inviscid medium, it is known that one cannot pose continuity of plane-parallel velocity. The only possible kinematic condition is the continuity of plane-normal velocity, which holds as long as there are no mixing or cavity forming between the two fluids

$$(\mathbf{u}\cdot\hat{\mathbf{n}})|_{z=0^-} = (\mathbf{u}\cdot\hat{\mathbf{n}})|_{z=0^+}. \tag{162}$$

In addition to the kinematic BC, Ferraro (1954) used the continuity of the magnetic field and the electric field for the electromagnetic boundary condition.

$$\mathbf{E}|_{z=0^-} = \mathbf{u}\times(\mathbf{B}_0+\mathbf{b})|_{z=0^-} = \mathbf{u}\times(\mathbf{B}_0+\mathbf{b})|_{z=0^+} = \mathbf{E}|_{z=0^+}, \quad \mathbf{b}|_{z=0^-} = \mathbf{b}|_{z=0^+} \tag{163}$$

[Compared to the insulating limit (at least we know vacuum is indeed an insulator), the ideal case seems to be harder to imagine, and it is dubious whether an ideal system would require the continuity of these fields. For instance, is an infinitely thin current sheet permitted in the infinitely conductive system? If that is the case, then magnetic field can afford discontinuity. However, continuity conditions on both magnetic and electric fields indeed close the equations appropriately in the end.] The continuity of electric field can be further manipulated by taking the cross product with $\hat{\mathbf{n}}$

$$\hat{\mathbf{n}}\times(\mathbf{u}\times(\mathbf{B}_0+\mathbf{b}))|_{z=0^-} = \hat{\mathbf{n}}\times(\mathbf{u}\times(\mathbf{B}_0+\mathbf{b}))|_{z=0^+}$$

$$(\hat{\mathbf{n}}\cdot(\mathbf{B_0}+\mathbf{b}))(\mathbf{u}|_{0^+}-\mathbf{u}|_{0^-}) = (\mathbf{B}_0+\mathbf{b})[(\mathbf{u}\cdot\hat{\mathbf{n}})|_{0^+}-(\mathbf{u}\cdot\hat{\mathbf{n}})|_{0^-}].$$

Due to the continuity of normal velocity, and considering $\mathbf{B}_0 \cdot \hat{\mathbf{n}} \neq 0$ (otherwise there is no Alfvén wave propagating towards the interface), we see that this amounts to requiring the velocity field to be continuous in all components across the boundary. The final boundary conditions are given by

$$\mathbf{u}|_{z=0^-} = \mathbf{u}|_{z=0^+}, \qquad \mathbf{b}|_{z=0^-} = \mathbf{b}|_{z=0^+}. \tag{164}$$

We seek solutions in the form of

$$\mathbf{b} = \begin{cases} \mathbf{b}^0 \exp\{i(\omega t - \mathbf{k}^0 \cdot \mathbf{r})\} + \mathbf{b}^R \exp\{i(\omega t - \mathbf{k}^R \cdot \mathbf{r})\}, & z < 0 \\ \mathbf{b}^T \exp\{i(\omega t - \mathbf{k}^T \cdot \mathbf{r})\}, & z > 0 \end{cases} \tag{165}$$

that satisfy the Alfvén wave equation and the aforementioned boundary conditions. R and T are superscripts for reflected and transmitted components, respectively. There are multiple ways to parameterize the wave vectors. For instance, Ferraro (1954) parameterized the wave vector by k and ω_0, which are quantities proportional to the projection of the wave vector onto $\hat{\mathbf{x}}'$ and $\hat{\mathbf{y}}$ axes, respectively. Although the parameterization allows effortless incorporation of the dispersion relation for Alfvén waves, thus gives simple expressions for waves in $Ox'yz'$ frame, the derivation of reflection and refraction laws is more tedious. In this article, however, I simply parameterize the wave vectors by the angles θ and ϕ (see remark box above). The wave vectors are hence given by

$$\begin{aligned} \mathbf{k}^0 &= k^0 \left(\sin\theta_0 \cos\phi_0, \sin\theta_0 \sin\phi_0, \cos\theta_0\right) \\ \mathbf{k}^R &= k^R \left(\sin\theta_R \cos\phi_R, \sin\theta_R \sin\phi_R, \cos\theta_R\right) \\ \mathbf{k}^T &= k^T \left(\sin\theta_T \cos\phi_T, \sin\theta_T \sin\phi_T, \cos\theta_T\right). \end{aligned} \tag{166}$$

The quantities k^0, k^R and k^T are magnitudes, and are hence positive; θ is defined in $[0, \pi]$, and $\phi \in [0, 2\pi)$. Note θ is defined as $< \mathbf{k}, \hat{\mathbf{z}} >$, and will be $> \pi/2$ for downgoing waves (i.e. phase velocity going to negative z direction).

A.1.2 Solutions for reflected and transmitted waves, laws of reflection and refraction

The parameterization and plane wave ansatz given above do not respect the Alfvén wave equation, nor the boundary condition. We first enforce the dispersion relation on the plane wave ansatz,

$$\mathbf{k}^0 \cdot \widehat{\mathbf{B}}_0 = \frac{\omega}{V_1}, \quad \mathbf{k}^R \cdot \widehat{\mathbf{B}}_0 = -\frac{\omega}{V_1}, \quad \mathbf{k}^T \cdot \widehat{\mathbf{B}}_0 = \frac{\omega}{V_2},$$

where $V_1 = B_0/\sqrt{\rho_1 \mu_0}$ and $V_2 = B_0/\sqrt{\rho_2 \mu_0}$. In this process we have already made the assumption that the incoming and transmitted wave propagates downwind of the background field ($\mathbf{k} \cdot \widehat{\mathbf{B}}_0 > 0$), while the reflected wave propagates upwind. This is justified by the fact that the incoming wave brings energy to the interface, while the reflected and transmitted waves bring energy away. Since [presumably] the energy propagates in the group velocity, it travels either in or opposite the background field. And since the background field is in the same direction as $\hat{\mathbf{n}}$, the downwind direction in medium 2 and upwind direction in medium 1 bring energy away from the interface.

Using this constraint, and noting $\widehat{\mathbf{B}}_0 = (\sin\beta, 0, \cos\beta)$, the magnitudes of the wave vectors can be solved as a function of θ and ϕ,

$$\begin{aligned} k^0 &= \frac{\omega}{V_1} \left(\sin\beta \sin\theta_0 \cos\phi_0 + \cos\beta \cos\theta_0\right)^{-1} \\ k^R &= -\frac{\omega}{V_1} \left(\sin\beta \sin\theta_R \cos\phi_R + \cos\beta \cos\theta_R\right)^{-1} \\ k^T &= \frac{\omega}{V_2} \left(\sin\beta \sin\theta_T \cos\phi_T + \cos\beta \cos\theta_T\right)^{-1} \end{aligned} \tag{167}$$

These solutions guarantee the plane wave solutions satisfy the dispersion relation, and hence satisfy the Alfvén wave equation. Next, we enforce the kinematic and electromagnetic boundary conditions at $z = 0$. This yields

$$\mathbf{b}^0 \exp\{i(\omega t - \mathbf{k}_H^0 \cdot \mathbf{r}) + \mathbf{b}^R \exp\{i(\omega t - \mathbf{k}_H^R \cdot \mathbf{r}) = \mathbf{b}^T \exp\{i(\omega t - \mathbf{k}_H^T \cdot \mathbf{r})\},$$

$$\frac{\mathbf{b}^0}{\sqrt{\rho_1}}\exp\{i(\omega t-\mathbf{k}_H^0\cdot\mathbf{r})-\frac{\mathbf{b}^R}{\sqrt{\rho_1}}\exp\{i(\omega t-\mathbf{k}_H^R\cdot\mathbf{r})=\frac{\mathbf{b}^T}{\sqrt{\rho_2}}\exp\{i(\omega t-\mathbf{k}_H^T\cdot\mathbf{r})\},$$

where $\mathbf{k}_H$ are the horizontal components of the wave vectors. The negative sign in the second equation comes due to the fact that waves propagating downwind and upwind have different phase differences between $\mathbf{b}$ and $\mathbf{u}$. These equations hold *if and only if* (i) the phases of the three wave match each other on the Oxy plane, which gives the phase equation

$$\mathbf{k}_H^0=\mathbf{k}_H^R=\mathbf{k}_H^T \tag{168}$$

and (ii) once the phases match, the amplitudes need to satisfy

$$\begin{aligned}\mathbf{b}^0+\mathbf{b}^R&=\mathbf{b}^T,\\ \frac{1}{\sqrt{\rho_1}}\left(\mathbf{b}^0-\mathbf{b}^R\right)&=\frac{1}{\sqrt{\rho_2}}\mathbf{b}^T.\end{aligned} \tag{169}$$

The amplitude equations simply yield the solutions

$$\mathbf{b}^R=\frac{\sqrt{\rho_2}-\sqrt{\rho_1}}{\sqrt{\rho_2}+\sqrt{\rho_1}}\mathbf{b}^0=\frac{V_1-V_2}{V_1+V_2}\mathbf{b}^0,\qquad \mathbf{b}^T=\frac{2\sqrt{\rho_2}}{\sqrt{\rho_2}+\sqrt{\rho_1}}\mathbf{b}^0=\frac{2V_1}{V_1+V_2}\mathbf{b}^0, \tag{170}$$

which give the reflection and refraction coefficients. These coefficients are independent of incidence angle, orientation of the background field, etc, but is only dependent on the media properties, which is different from seismic waves and Fresnel equations in optics. The phase equations, however, yields more complicated results than Snell's law

$$\begin{cases}k^0\sin\theta_0\cos\phi_0=k^R\sin\theta_R\cos\phi_R,\\ k^0\sin\theta_0\sin\phi_0=k^R\sin\theta_R\sin\phi_R,\end{cases}\qquad \begin{cases}\dfrac{k^0}{V_1}\sin\theta_0\cos\phi_0=\dfrac{k^T}{V_2}\sin\theta_T\cos\phi_T,\\ \dfrac{k^0}{V_1}\sin\theta_0\sin\phi_0=\dfrac{k^T}{V_2}\sin\theta_T\sin\phi_T.\end{cases} \tag{171}$$

Taking the ratio between the two equations within each bracket, we arrive at

$$\phi_0=\phi_R=\phi_T. \tag{172}$$

The line of logic is as follows. Taking the ratio yields $\tan\phi_0=\tan\phi_R=\tan\phi_T$, which contains an $n\pi$ ambiguity. However, since $k>0$, $\sin\theta>0$, for eq.(171) to hold it is further required that the cosines and sines of these angles have matching signs. Therefore it is only possible when all of these azimuths are the same. This is the *coplanar condition* for the reflection and transmission of Alfvén waves, and states that the incoming, reflected and refracted waves are propagating (in the sense of $\mathbf{k}$ or $\mathbf{c}_p$) in the same plane. Hereinafter I shall drop the subscript of ϕ since this quantity is shared by all waves. Using eq.(172) to further reduce the system, we arrive at

$$\frac{\sin\theta_0}{\sin\theta_R}=-\frac{\sin\beta\sin\theta_0\cos\phi+\cos\beta\cos\theta_0}{\sin\beta\sin\theta_R\cos\phi+\cos\beta\cos\theta_R},\qquad \frac{\sin\theta_0}{\sin\theta_T}=\frac{V_1}{V_2}\frac{\sin\beta\sin\theta_0\cos\phi+\cos\beta\cos\theta_0}{\sin\beta\sin\theta_T\cos\phi+\cos\beta\cos\theta_T}.$$

These two equations allow us to derive the expressions for θ_R and θ_T, as a function of β, θ_0 and ϕ.

$$\cot\theta_R+\cot\theta_0=-2\tan\beta\cos\phi, \tag{173}$$

which is the *law of reflection*, and

$$\cot\theta_T-\frac{V_1}{V_2}\cot\theta_0=\left(\frac{V_1}{V_2}-1\right)\tan\beta\cos\phi, \tag{174}$$

which is the *law of refraction*. The reflection and transmission coefficients (eq.170), the coplanar condition (eq.172) and the laws of reflection and refraction (eq.173-174) give all the information of the system. The plane wave solution, which originally has the form

$$\begin{aligned} \mathbf{b} &= \mathbf{b}^0 \exp\left\{ i\omega \left(t - \frac{1}{V_1} \frac{\sin\theta_0 \cos\phi x + \sin\theta_0 \sin\phi y + \cos\theta_0 z}{\sin\beta \sin\theta_0 \cos\phi + \cos\beta \cos\theta_0} \right) \right\} \\ &+ \mathbf{b}^R \exp\left\{ i\omega \left(t + \frac{1}{V_1} \frac{\sin\theta_R \cos\phi x + \sin\theta_R \sin\phi y + \cos\theta_R z}{\sin\beta \sin\theta_R \cos\phi + \cos\beta \cos\theta_R} \right) \right\}, \qquad z < 0 \\ \mathbf{b} &= \mathbf{b}^T \exp\left\{ i\omega \left(t - \frac{1}{V_2} \frac{\sin\theta_T \cos\phi x + \sin\theta_T \sin\phi y + \cos\theta_T z}{\sin\beta \sin\theta_T \cos\phi + \cos\beta \cos\theta_T} \right) \right\}, \qquad z > 0 \end{aligned} \tag{175}$$

will then become the following, after substituting the conditions,

$$\begin{aligned} \mathbf{b} &= \mathbf{b}^0 \exp\left\{ i\omega \left(t - \frac{1}{V_1} \frac{\sin\theta_0 \cos\phi x + \sin\theta_0 \sin\phi y + \cos\theta_0 z}{\sin\beta \sin\theta_0 \cos\phi + \cos\beta \cos\theta_0} \right) \right\} \\ &+ \frac{V_1 - V_2}{V_1 + V_2} \mathbf{b}^0 \exp\left\{ i\omega \left(t - \frac{1}{V_1} \frac{\sin\theta_0 \cos\phi x + \sin\theta_0 \sin\phi y + \sin\theta_0 \cot\theta_R z}{\sin\beta \sin\theta_0 \cos\phi + \cos\beta \cos\theta_0} \right) \right\}, \qquad z < 0 \\ \mathbf{b} &= \frac{2V_1}{V_1 + V_2} \mathbf{b}^0 \exp\left\{ i\omega \left(t - \frac{1}{V_1} \frac{\sin\theta_0 \cos\phi x + \sin\theta_0 \sin\phi y + \sin\theta_0 \cot\theta_T z}{\sin\beta \sin\theta_0 \cos\phi + \cos\beta \cos\theta_0} \right) \right\}, \qquad z > 0. \end{aligned} \tag{176}$$

As a side remark, one can easily shown this is equivalent to the plane wave solution in Ferraro 1954, which uses a different parameterization. The conversion is given in the following box.

Conversion to component parameterization

If one parameterizes the incoming wave as

$$\mathbf{b}^0 \exp\left\{ i \left(\omega t - \frac{\omega_1 x' + \omega_2 y + \omega z'}{V_1} \right) \right\} = \mathbf{b}^0 \exp\left\{ i\omega \left(t - \frac{(\frac{\omega_1}{\omega} \cos\beta + \sin\beta) x + \frac{\omega_2}{\omega} y + (-\frac{\omega_1}{\omega} \sin\beta + \cos\beta) z}{V_1} \right) \right\}$$

and introduces the horizontal frequency

$$\omega_H = \sqrt{(\omega_1 \cos\beta + \omega \sin\beta)^2 + \omega_2^2} = \frac{\omega \sin\theta_0}{\sin\beta \sin\theta_0 \cos\phi + \cos\beta \cos\theta_0},$$

one arrives at the expressions for the angles,

$$\cos\phi = \frac{1}{\omega_H} (\omega_1 \cos\beta + \omega \sin\beta), \quad \cot\theta_0 = \frac{1}{\omega_H} (-\omega_1 \sin\beta + \omega \cos\beta)$$

$$\cos\theta_R = -\frac{1}{\omega_H} \left(\omega_1 \sin\beta + \omega \frac{1 + \sin^2\beta}{\cos\beta} \right), \quad \cos\theta_T = \frac{1}{\omega_H} \left(-\omega_1 \sin\beta + \left(\frac{V_1}{V_2} - \sin^2\beta \right) \frac{\omega}{\cos\beta} \right)$$

Using these expressions and the laws of reflection and refraction, eq.(176) can be converted into the form

$$\begin{aligned} \mathbf{b} &= \mathbf{b}^0 \exp\left\{ i\omega t - \frac{i}{V_1} \left[(\omega_1 \cos\beta + \omega \sin\beta) x + \omega_2 y + (-\omega_1 \sin\beta + \omega \cos\beta) z \right] \right\} \\ &+ \mathbf{b}^R \exp\left\{ i\omega t - \frac{i}{V_1} \left[(\omega_1 \cos\beta + \omega \sin\beta) x + \omega_2 y - (\omega_1 \sin\beta + \omega \frac{1 + \sin^2\beta}{\cos\beta}) z \right] \right\}, \qquad z < 0 \\ \mathbf{b} &= \mathbf{b}^T \exp\left\{ i\omega t - \frac{i}{V_1} \left[(\omega_1 \cos\beta + \omega \sin\beta) x + \omega_2 y + \left(-\omega_1 \sin\beta + \left(\frac{V_1}{V_2} - \sin^2\beta \right) \frac{\omega}{\cos\beta} \right) z \right] \right\}, \qquad z > 0. \end{aligned} \tag{177}$$

This gives the exact same answer as Ferraro (1954).

A.1.3 Discussions on reflections and refractions

Let us recall the law of reflection (173) and the law of refraction (174). Except for the media properties, both laws are related to the (orientation of) background field via and *only* via the quantity

$$\tan\beta \cos\phi.$$

This is not just some random combination of the angles. Instead, let us recall the unit vectors of the background field and the axes in the frame of incidence

$$\widehat{\mathbf{B}}_0 = (\sin\beta, 0, \cos\beta), \quad \hat{\mathbf{x}}'' = (\cos\phi, \sin\phi, 0), \quad \hat{\mathbf{y}}'' = (\sin\phi, -\cos\phi, 0)$$

we have the identity

$$\tan\beta\cos\phi = \frac{\sin\beta\cos\phi}{\cos\beta} = \frac{\cos < \widehat{\mathbf{B}}_0, \hat{\mathbf{x}}'' >}{\cos < \widehat{\mathbf{B}}_0, \hat{\mathbf{z}} >}. \tag{178}$$

Therefore, this quantity is nothing but the ratio between two cosines of the angles formed by the background field with the rotated axis $Ox''y''z$. If this is still not straightforward enough, we can consider the projection of $\mathbf{B}_0$ onto the plane of incidence. Note that the plane normal of the plane of incidence is $\hat{\mathbf{y}}''$, the projection onto the plane of incidence can be given by

$$\begin{aligned}\widehat{\mathbf{B}}_0^{\|} &= \widehat{\mathbf{B}}_0 - (\widehat{\mathbf{B}}_0 \cdot \hat{\mathbf{y}})\hat{\mathbf{y}} = \sin\beta\hat{\mathbf{x}} + \cos\beta\hat{\mathbf{z}} - \sin\beta\sin\phi(\sin\phi\hat{\mathbf{x}} - \cos\phi\hat{\mathbf{y}}) \\ &= \sin\beta\cos^2\phi\hat{\mathbf{x}} + \sin\beta\sin\phi\cos\phi\hat{\mathbf{y}} + \cos\beta\hat{\mathbf{z}} \\ &= \sin\beta\cos^2\phi(\cos\phi\hat{\mathbf{x}}'' - \sin\phi\hat{\mathbf{y}}'') + \sin\beta\sin\phi\cos\phi(\sin\phi\hat{\mathbf{x}}'' + \cos\phi\hat{\mathbf{y}}'') + \cos\beta\hat{\mathbf{z}} \\ &= \sin\beta\cos\phi\hat{\mathbf{x}}'' + \cos\beta\hat{\mathbf{z}}\end{aligned} \tag{179}$$

Therefore the quantity can be written as

$$\tan\beta\cos\phi = \cot\langle\widehat{\mathbf{B}}_0^{\|}, \hat{\mathbf{x}}''\rangle \tag{180}$$

meaning this quantity is just the cotangent of the angle between the projected background field with the rotated $\hat{\mathbf{x}}''$ axis, or the tangent of the angle with the interface normal (with a variable sign). The value would be negative when the projection of the background field on the plane of incidence falls in the second quadrant of the $Ox''z$ plane. An easy way to achieve this is to have incidence wave coming from the other side, i.e. $\phi = \pi$, but keep the incidence relatively oblique so that the incidence wave still travels downwind. Hereinafter I shall denote $\alpha = \langle\mathbf{B}_0^{\|}, \hat{\mathbf{x}}''\rangle$.

The fact that the reflection and refraction laws only depend on this geometrical parameter greatly simplifies the geometrical setting; in other words, complicated as the multiple orientations seem, the only thing that matters is the projection of the background field in the plane of incidence. The component of $\mathbf{B}_0$ normal to the plane of incidence is irrelevant.

Both laws simplify significantly when the quantity is zero (Ferraro 1954). This corresponds to the scenario where the projection of background field on the plane of incidence is parallel to the interface normal. In particular, this happens when $\widehat{\mathbf{B}}_0$ is normal to the interface. The simplified laws in these scenarios are given by

$$\begin{aligned}\cot\theta_R = -\cot\theta_0 \quad &\Longrightarrow \quad \theta_R = \pi - \theta_0, \\ \cot\theta_T = \frac{V_1}{V_2}\cot\theta_0 \quad &\Longrightarrow \quad \frac{\tan\theta_T}{V_2} = \frac{\tan\theta_0}{V_1}\end{aligned} \tag{181}$$

so that the reflection angle is the same as the incidence angle (the $\pi - \theta_0$ just means it forms obtuse angle with the positive $\hat{\mathbf{z}}$ axis), and the refraction obeys a "Snell-like" law.

Finally, I comment on the criticality of the reflection and refraction angles. This is an essential and informative part, where we will see some counterintuitive results, and what separation of phase and group velocities can produce. For reflection,

$$\cot\theta_R = -\cot\theta_0 - 2\tan\beta\cos\phi = -\cot\theta_0 - 2\cot\alpha$$

I shall first assume that the incidence wave vector is in the same quadrant as $\widehat{\mathbf{B}}_0^{\|}$, so that $\theta_0 \in (0, \pi/2)$ and $\phi \in (-\pi/2, \pi/2)$. Although not explicitly stated, this is certainly an implicit assumption made when Ferraro discusses about the extrema of of reflection and refraction angles. In this case, the quantity $\cot\theta_R$ is guaranteed to be negative, and the reflection angle is guaranteed to be in the quadrant $(\pi/2, \pi)$.

As θ_0 goes from 0 (vertically downwards) to $\pi/2$ (easily verified that when $\phi \in (-\pi/2, \pi/2)$ all θ_0 in the first quadrant are valid incidence angles), θ_R goes from π (vertically downwards) to some critical angle

$$\theta_{R,\inf} = \pi - \operatorname{arccot}(2\tan\beta\cos\phi) = \pi - \operatorname{arccot}(2\cot\alpha) = \pi - \arctan\left(\frac{1}{2}\tan\alpha\right) \tag{182}$$

Therefore when ϕ is fixed, the phase propagation direction of the reflection cannot be arbitrarily close to the interface. For refraction, the situation is slightly more complicated, as

$$\cot\theta_T = \frac{V_1}{V_2}\cot\theta_0 + \left(\frac{V_1}{V_2} - 1\right)\tan\beta\cos\phi = \frac{V_1}{V_2}\cot\theta_0 + \left(\frac{V_1}{V_2} - 1\right)\cot\alpha$$

contains a coefficient with undetermined sign $V_1/V_2 - 1$. If the wave impinges on a denser medium, $\rho_2 > \rho_1$ so $V_2 < V_1$, the coefficient is positive, and θ_T is guaranteed to fall in quadrant $(0, \pi/2)$. As θ_0 increases from 0 to $\pi/2$, the refraction angle also monotonically increases from 0 to a limiting angle

$$\theta_{T,\sup} = \operatorname{arccot}\left(\frac{V_1 - V_2}{V_2}\tan\beta\cos\phi\right) = \operatorname{arccot}\left(\frac{V_1 - V_2}{V_2}\cot\alpha\right) = \arctan\left(\frac{V_2}{V_1 - V_2}\tan\alpha\right) \tag{183}$$

When the wave impinges on a lighter fluid, i.e. $\rho_2 < \rho_1$ so $V_1 < V_2$, the coefficient of the second term is negative. As θ_0 increases from 0 to a critical angle

$$\theta_{0,\text{crit}} = \operatorname{arccot}\left(\frac{V_2 - V_1}{V_1}\cot\ \alpha\right) = \arctan\left(\frac{V_1}{V_2 - V_1}\arctan\alpha\right), \tag{184}$$

the refraction angle θ_R increases from 0 to $\pi/2$, in which case the the phase of refracted wave seems to propagate along the surface. And as the incidence angle further increases to near $\pi/2$, the refracted wave begins to propagate downwards, up to an angle

$$\theta_{T,\sup} = \pi - \operatorname{arccot}\left(\frac{V_2 - V_1}{V_1}\cot\alpha\right) = \pi - \arctan\left(\frac{V_1}{V_2 - V_1}\tan\alpha\right). \tag{185}$$

The discussion on the same topic by Ferraro (1954) seems to be wrong, as in the process θ_T should actually be continuously changing, instead of going abruptly to π and then decrease. Either way, however, we see that according to these laws, the wave vector or the phase velocity of the transmitted wave can well be pointing downwards, i.e. $\mathbf{k}^T \cdot \hat{\mathbf{n}} < 0$ or $\mathbf{c}_p \cdot \hat{\mathbf{n}} < 0$. However, it can be easily shown that within this entire range, it remains true that $\mathbf{k}^T \cdot \mathbf{B}_0^{\parallel} = \mathbf{k}^T \cdot \mathbf{B}_0 > 0$. In fact, as long as the law of refraction holds, we can write

$$\begin{aligned}
\cos\langle\mathbf{k}^T, \mathbf{B}_0^{\parallel}\rangle &= (\cos\alpha\hat{\mathbf{x}}'' + \sin\alpha\hat{\mathbf{z}}) \cdot (\sin\theta_T\hat{\mathbf{x}}'' + \cos\theta_T\hat{\mathbf{z}}) \\
&= \cos\alpha\sin\theta_T + \sin\alpha\cos\theta_T \\
&= \frac{1}{\sin\theta_T\sin\alpha}(\cot\alpha + \cot\theta_T) \\
&= \frac{1}{\sin\theta_T\sin\alpha}\frac{V_1}{V_2}(\cot\alpha + \cot\theta_0) \\
&= \frac{V_1\sin\theta_0}{V_2\sin\theta_T}\cos\langle\mathbf{k}^0, \mathbf{B}_0^{\parallel}\rangle
\end{aligned}$$

Since θ_0 and θ_T are both defined within $[0, \pi]$, the sines are definitely non-negative. It follows if the incidence wave is propagating downwind ($\cos\langle\mathbf{k}^0, \mathbf{B}_0^{\parallel}\rangle > 0$), then the refracted wave is also guaranteed to propagate downwind. Therefore, the wave velocity still propagates in the positive $\mathbf{B}_0$ direction, carrying energy and information away from the interface.

We have now seen that the separation between phase and group velocities result in some counterintuitive results. When $\mathbf{B}_0$ is not perpendicular to the interface, a plane Alfvén wave can have wave vectors that are within $\pi/2$ from the $\widehat{\mathbf{B}}_0$, but $> \pi/2$ from $\hat{\mathbf{n}}$. The phase velocity, therefore, can point towards the interface, while the group velocity points away from it. For the incidence wave, it also means that it is possible that the phase velocity can point away from the interface, while the group velocity reserves a positive plane-normal component. This would correspond to a scenario where π falls in the second or third quadrant, i.e. $\in (\pi/2, 3\pi/2)$, and $\theta_0 > \pi/2$. As I have mentioned, most discussions above on the minimum and maximum implicitly assume $\theta_0 < \pi/2$ and $|\phi| < \pi/2$. [Once again I emphasize a comprehensive analysis of the energy flow is crucial in understanding the mechanism.]

Conservational quantities in reflection and refraction

In Snell's law for optics and seismology, we know that the horizontal slowness

$$p = \frac{\sin\theta}{V}$$

is preserved during reflection and refraction. I now realize one can derive similar quantities for Alfvén waves, where the laws of reflection and refraction can be written as

$$\cot\theta_R + \cot\alpha = -(\cot\theta_0 + \cot\alpha), \quad V_2(\cot\theta_T + \cot\alpha) = V_1(\cot\theta_0 + \cot\alpha)$$

and so the following quantity is preserved in reflection and refraction

$$V_A(\cot\theta + \cot\alpha) \tag{186}$$

where V_A takes the positive sign when wave propagates downwind, and the negative sign otherwise. We also further observe that

$$\hat{\mathbf{k}} \cdot \widehat{\mathbf{B}}_0^{\|} = \cos\alpha\sin\theta + \sin\alpha\cos\theta = \frac{\cot\theta + \cot\alpha}{\sin\theta\sin\alpha}$$

Therefore the quantity

$$V_A(\hat{\mathbf{k}} \cdot \widehat{\mathbf{B}}_0^{\|})\sin\theta\sin\alpha = c_p^{\|}\sin\theta\sin\alpha \tag{187}$$

is also preserved, where $c_p^{\|} = V_A(\hat{\mathbf{k}} \cdot \widehat{\mathbf{B}}_0^{\|})$ is the "phase velocity" constrained by the projected background field. The convention for the sign of V_A and c_p remains the same.

A.1.4 Polarization

The polarization in this problem is straightforward. From the solenoidal property of **B**, one concludes that

$$\mathbf{b}^0 \cdot \mathbf{k}_0 = 0, \quad \mathbf{b}^R \cdot \mathbf{k}_R = 0, \quad \mathbf{b}^T \cdot \mathbf{k}_T = 0$$

In addition, we have seen that under the aforementioned boundary conditions, all three waves (incidence, reflection, transmission) are polarized in the same direction. But since the three wave vectors are generally not aligned (except for the normal incidence case, where all polarizations are possible), the only possible polarization is

$$\mathbf{b}^0, \mathbf{b}^R, \mathbf{b}^T, \mathbf{u}^0, \mathbf{u}^R, \mathbf{u}^T \parallel \hat{\mathbf{y}}''. \tag{188}$$

[This raises yet another question: what would happen when the Alfvén waves polarized in directions other than $\hat{\mathbf{y}}''$ reaches the boundary? Remember that $\hat{\mathbf{y}}''$ is defined by the interface orientation and the wave vector. In the interior of the fluid, the wave does not "see" the boundary, and can of course polarize in any direction perpendicular to **k**.]

B Preliminary results of 3-D reflection

Here I document some intermediate, preliminary results that I derived when considering 3-D reflection. These results only apply to horizontally polarized Alfvén waves, indicating there might be ingredients missing from the picture.

B.1 Oblique incidence at resistive wall: inviscid case

First, I consider the oblique incidence at a solid resistive wall for inviscid fluid. For completely inviscid case, there is no Hartmann layer that can be established at the boundary.

B.1.1 Problem setup

From section (A.1), especially from eq.(180) and the related discussions, it seems it is more suitable to use the plane of incidence, instead of the plane of background field, as the main frame of reference. Inspired by this, I define Oxz as the plane of incidence. One can already postulate that the incidence and reflected waves share the same horizontal wavenumber, in which case $\phi_R = \phi_0 = 0$. The wave vectors of incidence and reflected waves are

$$\mathbf{k}^0 = k^0(\sin\theta_0, 0, \cos\theta_0), \quad \mathbf{k}^R = k^R(\sin\theta_R, 0, \cos\theta_R) \tag{189}$$

and the background field takes the form

$$\mathbf{B}_0 = B_0(\sin\theta_B\cos\phi_B, \sin\theta_B\sin\phi_B, \cos\theta_B) = (B_\parallel\cos\alpha, B_\perp, B_\parallel\sin\alpha). \tag{190}$$

As always, the background field is assumed to have positive vertical component, i.e. $\theta_B \in [0, \pi/2)$, or $\alpha \in (0, \pi)$. The whole fields in the fluids are

$$\begin{aligned}
\mathbf{b} &= \mathbf{b}^0 \exp\left\{i\omega\left[t - \frac{k_0}{\omega}(\sin\theta_0 x + \cos\theta_0 z)\right]\right\} + \mathbf{b}^R \exp\left\{i\omega\left[t - \frac{k_R}{\omega}(\sin\theta_R x + \cos\theta_R z)\right]\right\} \\
\mathbf{u} &= -\frac{\mathbf{b}^0}{\sqrt{\rho\mu_0}} \exp\left\{i\omega\left[t - \frac{k_0}{\omega}(\sin\theta_0 x + \cos\theta_0 z)\right]\right\} + \frac{\mathbf{b}^R}{\sqrt{\rho\mu_0}} \exp\left\{i\omega\left[t - \frac{k_R}{\omega}(\sin\theta_R x + \cos\theta_R z)\right]\right\}.
\end{aligned} \tag{191}$$

For the boundary condition, we require that the velocity field is non-penetrating (we don't yet have any assumption on the polarization), and that the electromagnetic field should match the electromagnetic field in the boundary. As before, we enforce homogeneous Dirichlet boundary condition on the magnetic field, effectively assuming that the field in the insulating wall is negligible. These boil down to

$$\hat{\mathbf{n}} \cdot \mathbf{u}|_{z=0} = 0, \qquad \mathbf{b}|_{z=0} = 0. \tag{192}$$

Naturally, there is no kinematic boundary condition for the other velocity components that can be enforced at the interface, due to the inviscid assumption.

B.1.2 Solution for the magnetic and velocity fields

In addition to the boundary conditions, we have the extra constraints that (i) the plane waves are solutions to the Alfvén wave equation, i.e. satisfy the aforementioned dispersion relation, and (ii) the plane waves represent longitudinal waves due to the solenoidal property of **u** and **b**. First, I use the dispersion relation to determine the wavenumbers. We recall that the dispersion relation can be transformed into

$$\frac{\mathbf{k}\cdot\mathbf{B}_0}{\sqrt{\rho\mu_0}} = \frac{\mathbf{k}\cdot\mathbf{B}_\parallel}{\sqrt{\rho\mu_0}} = \pm\omega \tag{193}$$

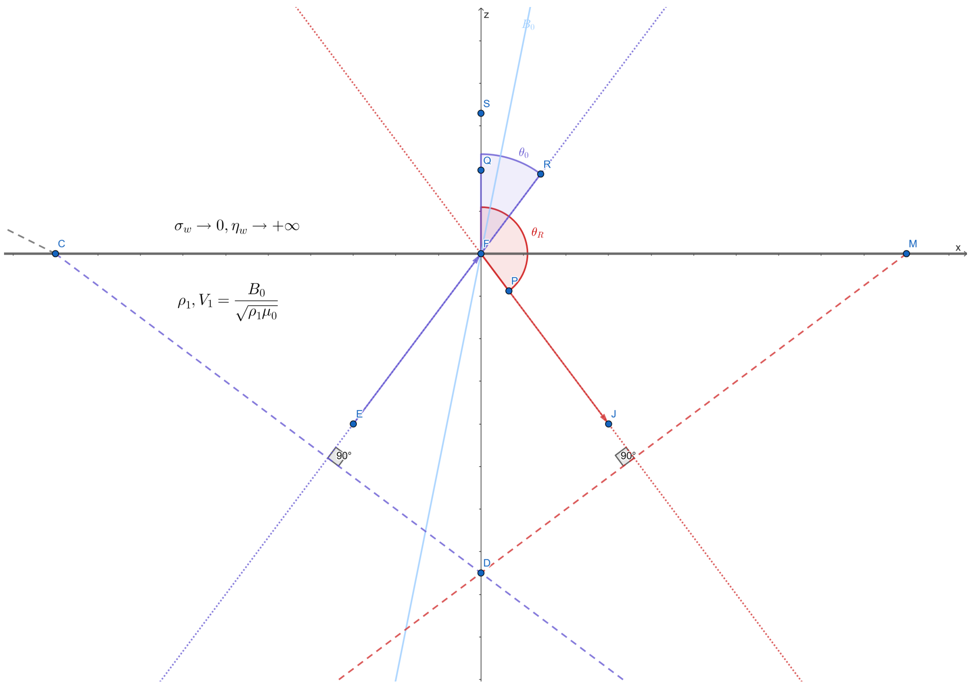

Figure 9: Setup of oblique incidence at resistive wall

where we have introduced $\mathbf{B}_{\|} = \mathbf{B}_0 - (\mathbf{B}_0 \cdot \hat{\mathbf{y}})\hat{\mathbf{y}}$, which is the projection of the background field onto the plane of incidence. If we redefine an Alfvén wave velocity that is induced by the in-plane component of background field

$$V_A^{\|} = \frac{B_{\|}}{\sqrt{\rho \mu_0}} \tag{194}$$

then for the travelling wave solutions, obviously we have

$$\begin{aligned} \mathbf{k}^0 \cdot \widehat{\mathbf{B}}_{\|} &= \frac{\omega}{V_A^{\|}}, \qquad k_0 = \frac{1}{\sin\theta_0 \cos\alpha + \cos\theta_0 \sin\alpha} \frac{\omega}{V_A^{\|}} = \frac{1}{\hat{\mathbf{k}}^0 \cdot \widehat{\mathbf{B}}_{\|}} \frac{\omega}{V_A^{\|}} \\ \mathbf{k}^R \cdot -\widehat{\mathbf{B}}_{\|} &= \frac{\omega}{V_A^{\|}}, \qquad k_0 = -\frac{1}{\sin\theta_R \cos\alpha + \cos\theta_R \sin\alpha} \frac{\omega}{V_A^{\|}} = -\frac{1}{\hat{\mathbf{k}}^R \cdot \widehat{\mathbf{B}}_{\|}} \frac{\omega}{V_A^{\|}} \end{aligned} \tag{195}$$

which gives the expression for the wave vectors. Next, we use the boundary conditions, which yield

$$\begin{aligned} &-(\hat{\mathbf{n}} \cdot \mathbf{b}^0) \exp\left\{-i\frac{\omega}{V_A^{\|}} \frac{\sin\theta_0 x}{\sin\theta_0 \cos\alpha + \cos\theta_0 \sin\alpha}\right\} + (\hat{\mathbf{n}} \cdot \mathbf{b}^R) \exp\left\{i\frac{\omega}{V_A^{\|}} \frac{\sin\theta_R x}{\sin\theta_R \cos\alpha + \cos\theta_R \sin\alpha}\right\} = 0 \\ &\mathbf{b}^0 \exp\left\{-i\frac{\omega}{V_A^{\|}} \frac{\sin\theta_0 x}{\sin\theta_0 \cos\alpha + \cos\theta_0 \sin\alpha}\right\} + \mathbf{b}^R \exp\left\{i\frac{\omega}{V_A^{\|}} \frac{\sin\theta_R x}{\sin\theta_R \cos\alpha + \cos\theta_R \sin\alpha}\right\} = 0 \end{aligned} \tag{196}$$

The matching of phase yields the relation

$$\begin{aligned} \frac{\sin\theta_0}{\sin\theta_0 \cos\alpha + \cos\theta_0 \sin\alpha} &= \frac{-\sin\theta_R}{\sin\theta_R \cos\alpha + \cos\theta_R \sin\alpha} \\ \implies \quad \cot\theta_R + \cot\alpha &= -\cot\theta_0 - \cot\alpha, \quad \cot\theta_R = -\cot\theta_0 - 2\cot\alpha \end{aligned} \tag{197}$$

which is the *law of reflection* for this setup. We see this is exactly the same as eq.(173). This does not come as a surprise; in general, all scenarios with similar settings will have the same reflection relation, as dictated by the matching of phase on the interface, or equivalently matching of horizontal wavenumber. The matching of amplitudes and the solenoidal property yield the relations

$$\hat{\mathbf{n}} \cdot (\mathbf{b}^0 - \mathbf{b}^R) = 0, \quad \mathbf{b}^0 + \mathbf{b}^R = \mathbf{0}, \quad \mathbf{b}^0 \cdot \mathbf{k}^0 = \mathbf{b}^R \cdot \mathbf{k}^R = 0. \tag{198}$$

The first two equation yields the relation

$$\mathbf{b}^R = -\mathbf{b}^0 \perp \hat{\mathbf{n}}, \tag{199}$$

which, in combination with the latter two relations, fixes the polarization of both the incoming and reflected Alfvén wave in the $\hat{\mathbf{y}}$ direction, or polarized normal to the plane of incidence. The reflection coefficient $R_b = -1$ indicates the half-wave loss when the Alfvén wave in an inviscid fluid impinges on an insulating wall.

B.1.3 Electromagnetic waves in the wall

Once again I check the role of electromagnetic wave solution in the insulating wall. To this end, the displacement current has to be reintroduced. **Here in both media I only consider the waves polarized in the $\hat{\mathbf{y}}$ direction, normal to the plane of incidence (the "AH" configuration).** In the insulating wall, the electromagnetic wave takes the form

$$\mathbf{b} = \hat{\mathbf{y}} b_w \exp\left\{ i\omega \left(t - \frac{1}{c}(\sin\theta_T x + \cos\theta_T z) \right) \right\}. \tag{200}$$

Now that we have a proper electromagnetic wave solution that couples the electric and magnetic fields in the insulating wall (note in the quasi-static limit where displacement current is neglected, the electric and the magnetic fields are virtually decoupled, and satisfy Laplace equation separately), we can then properly impose an additional continuity on the tangent electric field. To this end I shall first derive the electric fields in both media

$$\mathbf{e} = \begin{cases} \eta\nabla\times\mathbf{b} - \mathbf{u}\times\mathbf{B}_0 = \eta\nabla\times\mathbf{b} - u\hat{\mathbf{y}}\times\mathbf{B}_{\|} \\ \quad = \hat{\mathbf{x}}\left(i\eta k^0\cos\theta_0 b^0 e^{i\varphi_0} + i\eta k_R\cos\theta_R b^R e^{i\varphi_R}\right) - \hat{\mathbf{z}}\left(i\eta k^0\sin\theta_0 b^0 e^{i\varphi_0} + i\eta k^R\sin\theta_R b^R e^{i\varphi_R}\right) \\ \quad - \hat{\mathbf{x}}\frac{B_{\|}\sin\alpha}{\sqrt{\rho\mu_0}}\left(-b^0 e^{i\varphi_0} + b^R e^{i\varphi_R}\right) + \hat{\mathbf{z}}\frac{B_{\|}\cos\alpha}{\sqrt{\rho\mu_0}}\left(-b^0 e^{i\varphi_0} + b^R e^{i\varphi_R}\right) \\ \quad = \hat{\mathbf{x}}\left(\left(i\eta k^0\cos\theta_0 + V_A^{\|}\sin\alpha\right) b^0 e^{i\varphi_0} + \left(i\eta k_R\cos\theta_R - V_A^{\|}\sin\alpha\right) b^R e^{i\varphi_R}\right) \\ \quad - \hat{\mathbf{z}}\left(\left(i\eta k_0\sin\theta_0 + V_A^{\|}\cos\alpha\right) b^0 e^{i\varphi_0} + \left(i\eta k_R\sin\theta_R - V_A^{\|}\cos\alpha\right) b^R e^{i\varphi_R}\right), \quad (z<0) \\ \frac{c^2}{i\omega}\nabla\times\mathbf{b} = \left(\hat{\mathbf{x}}c\cos\theta_T - \hat{\mathbf{z}}c\sin\theta_T\right) b_w e^{i\varphi_T}, \quad (z>0) \end{cases} \tag{201}$$

We see that the electric field is polarized in the plane of incidence, and has in general both $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components. In the insulating medium, the electric field is in phase with the magnetic field, characteristic of the electromagnetic wave in vacuum. It is also the case for the electric field induced by fluid motion in the conducting fluid. The conducted electric field, however, shows a $\pi/2$ phase difference.

Now we are in the position to impose two electromagnetic boundary conditions: continuity of the tangent magnetic field, and the continuity of the tangent electric field

$$b_y|_{z=0^-} = b_y|_{z=0^+}, \quad e_x|_{z=0^-} = e_x|_{z=0^+}.$$

The two relations give

$$\begin{gathered} b^0 e^{i\varphi_0} + b^R e^{i\varphi_R} = b_w e^{i\varphi_T} \\ \left(i\eta k^0\cos\theta_0 + V_A^{\|}\sin\alpha\right) b^0 e^{i\varphi_0} + \left(i\eta k^R\cos\theta_R - V_A^{\|}\sin\alpha\right) b^R e^{i\varphi_R} = c\cos\theta_T b_w e^{i\varphi_T} \end{gathered} \tag{202}$$

Again this requires the matching of both phase and amplitudes. The phase matching gives

$$\frac{\sin\theta_0}{\sin\theta_0\cos\alpha + \cos\theta_0\sin\alpha} = \frac{-\sin\theta_R}{\sin\theta_R\cos\alpha + \cos\theta_R\sin\alpha} = \frac{V_A^{\|}}{c}\sin\theta_T \tag{203}$$

Aside from the already established reflection law (eq.197), we also obtain the law of refraction

$$\sin\theta_T = \frac{c}{V_A^{\|}}\frac{\tan\theta_0}{\tan\theta_0\cos\alpha+\sin\alpha} = \beta_A^{-1}\frac{\tan\theta_0}{\tan\theta_0\cos\alpha+\sin\alpha} = \beta_A^{-1}\frac{1}{\cos\alpha+\cot\theta_0\sin\alpha} \tag{204}$$

where we once again introduced the dimensionless group $\beta_A = V_A^{\|}/c$ defined for the in-plane Alfvén wave velocity. At the limit of normal incidence, $\theta_0 \to 0$, it is apparent that $\sin\theta_T \to 0$ and so the transmitted wave is also normal to the boundary. However, because $\beta_A \sim 10^{-11}$ in the Earth core is very small, only within a very narrow range of θ_0 ($\theta_0 \sim 10^{-11}$rad if $\tan\alpha \sim 1$) can the RHS be no greater than 1 for the transmitted electromagnetic wave to be a travelling wave. Otherwise, $\sin\theta_T$ will be massively greater than 1, which gives rise to a purely imaginary $\cos\theta_T$, forcing the transmitted wave to be a evanescent wave in the z direction, which takes the form

$$b_w \exp\left\{i\omega t - \frac{\omega}{V_A^{\|}}\frac{ix+\sqrt{1-\beta_A^2(\cos\alpha+\cot\theta_0\sin\alpha)^2}z}{\cos\alpha+\cot\theta_0\sin\alpha}\right\} \approx b_w \exp\left\{i\omega t - \frac{\omega}{V_A^{\|}}\frac{ix+z}{\cos\alpha+\cot\theta_0\sin\alpha}\right\}$$

when $|\beta_A(\cos\alpha+\cot\theta_0\sin\alpha)| \ll 1$. The "skin-depth", or the length scale over which the wave decays, is $V_A^{\|}/\omega$, which is the Alfvén wave length λ_A. Therefore, even if the wave is evanescent, the skin depth can be considerable, especially at lower periods.

Assuming the phases are properly matched, and plugging in the expression for k, the second equation can be rearranged into

$$\left(i\frac{\omega\eta}{V_A^{\|}}\frac{1}{\tan\theta_0\cos\alpha+\sin\alpha}+V_A^{\|}\sin\alpha\right)b^0 - \left(i\frac{\omega\eta}{V_A^{\|}}\frac{1}{\tan\theta_0\cos\alpha+\sin\alpha}+V_A^{\|}\sin\alpha\right)b^R = c\cos\theta_T b_w$$

Introducing the Lundquist number for in-plane background field

$$S_{\eta\|} = \frac{\left(V_A^{\|}\right)^2}{\omega\eta}$$

and the set of equations are reduced to

$$\begin{gathered} b^0 + b^R = b_w \\ \beta_A\left[\left(iS_{\eta\|}^{-1}\frac{1}{\tan\theta_0\cos\alpha+\sin\alpha}+\sin\alpha\right)b^0 - \left(iS_{\eta\|}^{-1}\frac{1}{\tan\theta_R\cos\alpha+\sin\alpha}+\sin\alpha\right)b^R\right] = \cos\theta_T b_w \end{gathered} \tag{205}$$

The role of transmitted wave is again seen as a higher order correction in β_A for the reflection coefficients

$$R_b = \frac{b^R}{b^0} = -\frac{\cos\theta_T - \beta_A\left(\sin\alpha + iS_{\eta\|}^{-1}\frac{1}{\tan\theta_0\cos\alpha+\sin\alpha}\right)}{\cos\theta_T + \beta_A\left(\sin\alpha + iS_{\eta\|}^{-1}\frac{1}{\tan\theta_R\cos\alpha+\sin\alpha}\right)} \tag{206}$$

At $\beta_A \to 0$ it simply reduces to the half-wave loss $R_b = -1$. For normal incidence with background field normal to the plane, $\alpha = \theta_0 = \theta_T = 0$ and $\theta_R = \pi$, the relation reduces to

$$R_b = -\frac{1-\beta_A(1+S_\eta^{-1})}{1+\beta_A(1+S_\eta^{-1})}$$

which gives exactly the same relation as the limiting case at Pm $\to 0$ for the 1-D model.

[Similar to the Ferraro (1954) case, the inviscid case seems to only permit reflection and refraction of waves polarized in a specific direction, i.e. normal to the plane of incidence (the *horizontally polarized Alfvén wave*). The natural question is, what happens to Alfvén waves with other polarizations? In seismology, the so-called SV waves can be converted to P waves at the boundary. However, there is no analog of P wave in the context of Alfvén waves. Therefore, with vertical polarization, Alfvén waves (and EM waves) cannot simultaneously satisfy the magnetic boundary condition and the non-penetration boundary condition. However, a vertically polarized Alfvén wave is perfectly imaginable and physically possible. What is the way out of this?]

[There might three candidates out of this. (i) Viscous dissipation can be reintroduced to build up some boundary layers (e.g. Hartmann boundary layer, etc.) Problem is, anisotropic boundary layer does not seem to allow oscillation in one direction ($\mathbf{x}$) while decaying in another, this means there is little hope to match the horizontal wavenumber. (ii) Incompressibility of the fluid may be relaxed; one may require other kinds of waves to be triggered at the boundary, for instance the magneto-acoustic waves. However, the wave speeds and length scales may be very different. (iii) The forces at the boundary is accounted for by something other than waves. The Alfvén wave equation might be inadequate.]

B.2 Oblique incidence at resistive wall: viscous case

I hereby carry out the oblique incidence analysis in a regime where both magnetic and viscous diffusions are present. To this end, I shall make use of the results from the previous section, where I explicitly derived the expression for $\tilde{k}_z$ when the horizontal slowness $\bar{p}$ is given beforehand. I shall also adopt a parameterization in line with these derivations. Instead of parameterizing the wave vector as $\mathbf{k} = k(\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{z}})$, I shall parameterize it as $\mathbf{k} = \omega p\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}$. The fields can then be written as

$$\begin{aligned}
\mathbf{b} &= \mathbf{b}^0 \exp\left\{i\left(\omega t - \omega p x - k_z^0 z\right)\right\} + \mathbf{b}^R \exp\left\{i\left(\omega t - \omega p x - k_z^R z\right)\right\} \\
&\quad + \mathbf{b}^{\mathrm{BL}} \exp\left\{i\left(\omega t - \omega p x - k_z^{\mathrm{BL}} z\right)\right\} \\
\mathbf{u} &= C_0 \frac{\mathbf{b}^0}{\sqrt{\rho\mu_0}} \exp\left\{i\left(\omega t - \omega p x - k_z^0 z\right)\right\} + C_R \frac{\mathbf{b}^R}{\sqrt{\rho\mu_0}} \exp\left\{i\left(\omega t - \omega p x - k_z^R z\right)\right\} \\
&\quad + C_{\mathrm{BL}} \frac{\mathbf{b}^{\mathrm{BL}}}{\sqrt{\rho\mu_0}} \exp\left\{i\left(\omega t - \omega p x - k_z^{\mathrm{BL}} z\right)\right\}
\end{aligned} \tag{207}$$

The merit of this formulation is that the phases are automatically matched. For k_z, we need to use the results from the previous section.

B.2.1 Zeroth-order approximation

Dimensionalizing the expressions for $\tilde{k}_z$, we have

$$\begin{aligned}
\tilde{k}_z^{\mathrm{prop}} &= \frac{\pm 1 - \bar{p}\cos\alpha}{\sin\alpha} \quad \Longrightarrow \quad k_z^{\mathrm{prop}} = \frac{\omega}{V_A \sin\alpha}\left(\pm 1 - pV_A\cos\alpha\right) \\
\tilde{k}_z^{\mathrm{BL}} &= \pm i \frac{S_\eta \sin\alpha}{\sqrt{\mathrm{Pm}}} \quad \Longrightarrow \quad k_z^{\mathrm{BL}} = \pm i \frac{V_A \sin\alpha}{\sqrt{\nu\eta}}
\end{aligned} \tag{208}$$

Here I use the zeroth-order approximation, because it is the only approximation I have for arbitrary α. Note that what Schaeffer, Jault, et al. (2012) and Schaeffer and Jault (2016) used is also the zeroth-order approximation. The relative error in their wavenumbers are also $O(S_\omega^{-1})$. For the incidence wave, the positive sign in k_z^{prop} is taken; for the reflected wave, we take the negative sign; for the Hartmann layer, we take the positive sign because we need a solution that decays in the negative z-direction. In the end, the magnetic field is

$$\begin{aligned}
\mathbf{b} &= \mathbf{b}^0 \exp\left\{i\omega\left(t - px - \frac{1 - pV_A\cos\alpha}{V_A\sin\alpha}z\right)\right\} \\
&\quad + \mathbf{b}^R \exp\left\{i\omega\left(t - px + \frac{1 + pV_A\cos\alpha}{V_A\sin\alpha}z\right)\right\} \\
&\quad + \mathbf{b}^{\mathrm{BL}} \exp\left\{i\omega\left(t - px\right) + \frac{V_A\sin\alpha}{\sqrt{\nu\eta}}z\right\}.
\end{aligned} \tag{209}$$

To the zeroth order, we see from below that there are no modifications as to the coefficients of the velocity field. The derivations should still be based on the Navier-Stokes part of the Alfvén wave equation

$$(i\omega + \nu k^2)\mathbf{u} = -i\frac{\mathbf{B}_0 \cdot \mathbf{k}}{\rho\mu_0}\mathbf{b} = -iV_A(\widehat{\mathbf{B}}_0 \cdot \mathbf{k})\frac{\mathbf{b}}{\sqrt{\rho\mu_0}}$$

For the travelling wave, $\nu k^2 \ll \omega$ and $\widehat{\mathbf{B}}_0 \cdot \mathbf{k} = \omega/V_A$ we have

$$\mathbf{u} = \mp\frac{\mathbf{b}}{\sqrt{\rho\mu_0}} + O\left(S_\eta^{-1}\right).$$

For the boundary layer solution, again we have $\nu k^2 \gg \omega$; the inner product $\widehat{\mathbf{B}}_0 \cdot \mathbf{k}$ is to zeroth order $iV_A \sin\alpha/\sqrt{\nu\eta}$. The $\sin^2\alpha$ that arises from k^2 should in the end cancel out with the same factor from $\widehat{\mathbf{B}}_0 \cdot \mathbf{k}$. Therefore, the velocity field is still

$$\mathbf{u} = -\mathrm{Pm}^{-\frac{1}{2}}\frac{\mathbf{b}}{\sqrt{\rho\mu_0}} + O\left(S_\eta^{-1}\right).$$

In the end, $C_0 = -1$. $C_R = 1$, $C_{\mathrm{BL}} = -\mathrm{Pm}^{-\frac{1}{2}}$. The velocity field can still be written as

$$\begin{aligned}\mathbf{u} = &-\frac{\mathbf{b}^0}{\sqrt{\rho\mu_0}}\exp\left\{i\omega\left(t - px - \frac{1 - pV_A\cos\alpha}{V_A\sin\alpha}z\right)\right\}\\ &+\frac{\mathbf{b}^R}{\sqrt{\rho\mu_0}}\exp\left\{i\omega\left(t - px + \frac{1 + pV_A\cos\alpha}{V_A\sin\alpha}z\right)\right\}\\ &-\frac{1}{\sqrt{\mathrm{Pm}}}\frac{\mathbf{b}^{\mathrm{BL}}}{\sqrt{\rho\mu_0}}\exp\left\{i\omega\left(t - px\right) + \frac{V_A}{\sqrt{\nu\eta}}z\right\}.\end{aligned} \tag{210}$$

For the boundary condition, in the viscous case we should have both continuity of magnetic field (which gives homogeneous Dirichlet condition on $\mathbf{b}$), as well as the continuity of velocity (which gives no-slip condition). Setting both fields to be zero at the boundary, we end up with the exact same relation as before

$$\begin{aligned}\mathbf{b}^0 + \mathbf{b}^R + \mathbf{b}^{\mathrm{BL}} &= \mathbf{0}\\ -\mathbf{b}^0 + \mathbf{b}^R - \mathrm{Pm}^{-\frac{1}{2}}\mathbf{b}^{\mathrm{BL}} &= \mathbf{0}\end{aligned} \tag{211}$$

which gives the relations

$$\mathbf{b}^{\mathrm{BL}} = -\frac{2\sqrt{\mathrm{Pm}}}{1 + \sqrt{\mathrm{Pm}}}\mathbf{b}^0, \quad \mathbf{b}^R = -\frac{1 - \sqrt{\mathrm{Pm}}}{1 + \sqrt{\mathrm{Pm}}}\mathbf{b}^0. \tag{212}$$

These are exactly the same relation as in the 1-D normal incidence case. It seems that, along the derivations, nothing changes. While it is straightforward to see *a priori* that the boundary condition on magnetic field yields the same equation, the boundary condition on the velocity field also does not introduce anything new. Indeed, when pV_A is of order unity and so is $\sin\alpha$, to first order accuracy in S_η^{-1}, the ratio between the $\mathbf{u}$ and $\mathbf{b}$ is independent of both the orientation of the background magnetic field and the horizontal wavenumber. However, I do anticipate the orientation quantities to be present in first order refinement term (of the order S_η^{-1}).

Compared to the 1-D problem, there are only two new things in the oblique incidence. First, the reflection law is non-trivial. Although one can independently derive this law of reflection in this parameterization, there is no doubt the result takes the exact same form as eq.(197) as well as eq.(173). The reason is that all that matters in the law of reflection (and refraction) is the match of argument of the exponential function, which remains the same for all oblique incidence problems involving incidence and reflected planar Alfvén waves.

Secondly, since in principle eq.212 give vector equations, we can conclude that the three solutions (incidental Alfvén wave, reflected Alfvén wave and Hartmann layer) are polarized in the same direction. Furthermore, due to the solenoidal condition on $\mathbf{u}$ and $\mathbf{b}$, this polarization should be perpendicular to all wave vectors. From this we again arrive at the same conclusion as Ferraro (1954), that the incidental and reflected Alfvén waves (in this case also the Hartmann layer) are polarized in the y-direction, i.e. in and out of the plane of incidence.

[It remains unsolved, however, how vertically-polarized Alfvén wave should behave at the boundaries.]

B.2.2 Precise form

Here I list the corresponding precise form in terms of k_z, where k_z needs to be calculated using eq.(68) or (70). The coefficients C_0, C_R and $C_{\rm BL}$ are given by

$$C = \frac{-iV_A(\omega p \cos\alpha + k_z \sin\alpha)}{i\omega + \nu(\omega^2 p^2 + k_z^2)} \tag{213}$$

where k_z takes the respective solutions from eq.(70) for each wave. The boundary condition then yields

$$\begin{aligned} \mathbf{b}^0 + \mathbf{b}^R + \mathbf{b}^{\rm BL} &= 0 \\ C_0\mathbf{b}^0 + C_R\mathbf{b}^R + C_{\rm BL}\mathbf{b}^{\rm BL} &= 0. \end{aligned} \tag{214}$$

The solutions are then given by

$$\mathbf{b}^{\rm BL} = -\frac{C_R - C_0}{C_R - C_{\rm BL}}\mathbf{b}^0, \quad \mathbf{b}^{\rm BL} = -\frac{C_0 - C_{\rm BL}}{C_R - C_{\rm BL}}\mathbf{b}^0. \tag{215}$$

Since the coefficients C are in general functions in the form of $C(\omega, p, \alpha)$, one can anticipate the reflection coefficients will also be a function of these quantities.

C Solving perturbed eigenvalue problems

C.1 Perturbed polynomial equations

In this section, I shall discuss the modification of roots of polynomials in presence of perturbations in the coefficients. This arises when a polynomial (arising e.g. as characteristic polynomials in eigenvalue problems) is not analytically solvable, but a simplified version (e.g. when neglecting some small dimensionless quantities) has known solutions. It is then of interest to know how the solutions of the polynomial are modified in presence of the small quantities as perturbations.

Let us consider an arbitrary degree-N polynomial equation

$$p(x) = \sum_{n=0}^{N} a_n x^n = 0 \tag{216}$$

whose roots are known and given by $\{x_{0,n}\}_{n=1:N}$. Now, let us consider the *perturbed* equation

$$p^*(x) = p(x) + \epsilon\delta p(x) = \sum_{n=0}^{N} (a_n + \epsilon\delta a_n) x^n = 0 \tag{217}$$

where δa_n is the perturbation term and ϵ gives the perturbation amplitude. $\delta p(x)$ is defined as $\sum_n \delta a_n x^n$. The roots to this polynomial are unknown, and I denote these as $\{x_n^*\}$. I take this form so that when $\epsilon \to 0$, $x_n^* \to x_{0,n}$. To quantify the sensitivity to the perturbation, I follow closely how one would normally derive a condition number for numerical methods, and write $x^* = x_0 + \delta x$,

$$\sum_{n=0}^{N} (a_n + \epsilon\delta a_n)(x_0 + \delta x)^n = 0. \tag{218}$$

Of course, this is still a degree-N polynomial of δx. However, as ϵ can be arbitrarily small, so is δx, and the polynomial can be kept to the leading order. Although it is unclear what is the "leading order", since we have no prior information on the scaling between δx and ϵ, one scenario is particularly simple. When the root x_0 is a simple root, and so $\frac{d}{dx}\left(\sum_n a_n x^n\right)(x = x_0) \neq 0$, we can simply linearize the equation,

$$\begin{aligned}
&\sum_{n=0}^{N} (a_n + \epsilon\delta a_n)(x_0 + \delta x)^n \\
\approx &\sum_{n=0}^{N} (a_n + \epsilon\delta a_n)\left(x_0^n + n x_0^{n-1}\delta x\right) \\
\approx &\sum_{n=0}^{N} a_n x_0^n + \sum_{n=0}^{N} n a_n x_0^{n-1}\delta x + \epsilon \sum_{n=0}^{N} \delta a_n x_0^n \\
= &\,\delta x \sum_{n=0}^{N} n a_n x_0^{n-1} + \epsilon \sum_{n=0}^{N} \delta a_n x_0^n = 0,
\end{aligned}$$

which yields the perturbation in the root

$$\delta x \approx -\epsilon \frac{\sum_{n=0}^{N} \delta a_n x_0^n}{\sum_{n=0}^{N} n a_n x_0^{n-1}} = -\epsilon \frac{\delta p(x_0)}{p'(x_0)}. \tag{219}$$

This result not only gives the leading order perturbation in the root, but also shows that $\delta x \sim \epsilon$ to the leading order (unless, of course, $\delta p(x_0) = 0$, in which case x_0 remains the true solution to the perturbed system).

While the previous equation applies to any simple root, it does not hold for any multiple root. The most obvious reason is that as $p'(x) = 0$ at multiple roots, the linear term from the approximation is trivial, and eq.(219) is no longer valid. Conversely, one can prove that δx associated with roots x_0 with multiplicity cannot have the asymptotic behaviour of $O(\epsilon)$. It in turn indicates that linearization fails to capture all necessary terms of leading order. For a root with multiplicity of K, it is necessary to keep all terms up to δx^K,

$$\begin{aligned}
&\sum_{n=0}^{N}(a_n+\epsilon\delta a_n)\sum_{k=0}^{n}\binom{n}{k}x_0^{n-k}\delta x^k \\
\approx&\sum_{k=0}^{K}\delta x^k\sum_{n=k}^{N}\binom{n}{k}(a_n+\epsilon\delta a_n)x_0^{n-k} \\
=&\sum_{k=0}^{K}\delta x^k\left[\sum_{n=k}^{N}\binom{n}{k}a_nx_0^{n-k}+\epsilon\sum_{n=k}^{N}\binom{n}{k}\delta a_nx_0^{n-k}\right] \\
=&\sum_{k=0}^{K}\frac{\delta x^k}{k!}p^{(k)}(x_0)+\epsilon\sum_{k=0}^{K}\frac{\delta x^k}{k!}\delta p^{(k)}(x_0)
\end{aligned}$$

Since the solution x_0 has multiplicity K, we have $p^{(k)}(x_0)=0$ $(\forall k<K)$. Only one term in the first summation is kept. Therefore, the equation is simplified

$$\begin{aligned}
p^{(K)}(x_0)\frac{\delta x^K}{K!}+\epsilon\sum_{k=0}^{K}\delta p^{(k)}(x_0)\frac{\delta x^k}{k!}&=0 \\
\delta x^K\sum_{n=K}^{N}\binom{n}{K}a_nx_0^{n-k}+\epsilon\sum_{k=0}^{K}\delta x^k\sum_{n=k}^{K}\binom{n}{k}\delta a_nx_0^{n-k}&=0.
\end{aligned} \tag{220}$$

Note that since the asymptotic behaviour of δx at $\epsilon\to 0$ is unknown, in generall all of the terms in the second summation need to be kept. Therefore, for a solution with multiplicity K, this approach still requires solving a degree-K equation in general. The K solutions to eqn.(220) create the splitting of the original K-fold multiple roots, unless there are further multiplicity in the perturbations δx, in which case even higher powers need to be kept.

As a proof of concept, we look at two very simple, low-degree examples, whose closed-form solutions are easily calculated. For the simple root example, let us look at

$$p^*(x)=x^2-(1+\epsilon)=\left(x^2-1\right)-\epsilon=p(x)+\epsilon\delta p(x).$$

The unperturbed polynomial $p(x)$ has two simple roots $x_{1,2}=\pm 1$. Following eqn.(219), the respective perturbations are given by

$$\delta x_{1,2}\approx-\epsilon\frac{\delta p(x_{1,2})}{p'(x_{1,2})}=-\epsilon\frac{-1}{\pm 2}=\pm\frac{\epsilon}{2}$$

On the other hand, the exact solution to the perturbed system is known, and can be expanded as Taylor series of ϵ at $\epsilon=0$,

$$x^*_{1,2}=\pm\sqrt{1+\epsilon}\approx\pm 1\pm\frac{\epsilon}{2}+O(\epsilon^2)$$

whose linear term is consistent with $\delta x_{1,2}$. For the multiple root example, we look at

$$p^*(x)=x^2-(2+2\epsilon)x+1=(x-1)^2-\epsilon(2x)=p(x)+\epsilon\delta p(x).$$

The unperturbed polynomial has a double root $x_1=x_2=1$. Following eqn.(220), we keep the approximation to $K=2$, and get

$$\frac{2}{2}\delta x^2+\epsilon\left(-2-2\delta x\right)=\delta x^2-2\epsilon\delta x-2\epsilon=0$$

which gives

$$\delta x_{1,2}=\epsilon\pm\sqrt{\epsilon(2+\epsilon)}$$

In fact, this gives the exact solution of the perturbed system,

$$x_{1,2}^* = 1 + \epsilon \pm \sqrt{\epsilon(2+\epsilon)}.$$

The double root example indicates another complication of estimating δx in terms of ϵ. In this example, the asymptotic behaviour of δx at $\epsilon \longrightarrow 0$ is $\sim \sqrt{2\epsilon}$. This asymptotic function has an essential singularity (branch point in the complex plane) at $\epsilon \longrightarrow 0$, and therefore has no power series (Taylor or Laurent) at small perturbations. It would hence be a lost cause to expand δx in terms of ϵ.

C.2 Perturbed eigenvalue problems

While the previous section provides a neat way to calculated the perturbed eigenvalue from the characteristic polynomial, it is often (as is the case of the derivations in previous sections) of equal interest to know the perturbation in the eigenvectors. There are two approaches that can be taken. First, one can plug the eigenvalue (with the correction) back into the original system to obtain the corrections on the eigenvectors. For instance, the relative ratio between magnetic and velocity field in the Alfvén mode can be retrieved from the frequency-wavenumber-domain induction equation

$$\left(i\omega + \eta k^2\right)\mathbf{b} = -iB_0 k_z \mathbf{u}.$$

On the other hand, the corrections in the eigenvalue and the eigenvectors can also be simultaneously obtained by solving the perturbed eigenvalue problem. Let us consider the *unperturbed* matrix eigenvalue problem

$$\mathbf{K}_0\mathbf{x} = \lambda\mathbf{M}_0\mathbf{x} \tag{221}$$

whose eigenvalues, denoted as $\{\lambda_{0n}\}_{n=1:N}$, and the corresponding eigenvectors $\{\mathbf{x}_{0n}\}_{n=1:N}$, are both known. Let us further assume that $\mathbf{K}_0$ and $\mathbf{M}_0$ are Hermitian (self-adjoint) matrices, and the eigenvalues are all distinct. This leads to the orthogonality

$$\mathbf{x}_{0i}^H\mathbf{M}_0\mathbf{x}_{0j} = \delta_{ij}\mathbf{x}_{0i}^H\mathbf{M}_0\mathbf{x}_{0i} = \delta_{ij}\|\mathbf{x}_{0i}\|_{\mathbf{M}_0}^2 = \delta_{ij}. \tag{222}$$

where the vector norms of the eigenvectors are assumed to be one to give unique eigenvectors (except for a sign difference). Now consider the perturbed system

$$\mathbf{K}\mathbf{x} = \lambda\mathbf{M}\mathbf{x}, \qquad \mathbf{K} = \mathbf{K}_0 + \delta\mathbf{K}, \quad \mathbf{M} = \mathbf{M}_0 + \delta\mathbf{M} \tag{223}$$

where the Hermitian property is conserved. The eigenvalues and corresponding eigenvectors are denoted as $\{\lambda_n^*\}_{n=1:N} = \{\lambda_{0n} + \delta\lambda_n\}$ and $\{\mathbf{x}_n^*\}_{n=1:N} = \{\mathbf{x}_{0n} + \delta\mathbf{x}_n\}$, respectively. From the orthogonality condition we have to leading order of perturbation

$$(\mathbf{x}_{0i} + \delta x_i)^H (\mathbf{M}_0 + \delta\mathbf{M}) \left(\mathbf{x}_{0j} + \delta\mathbf{x}_j\right) \approx \mathbf{x}_{0i}^H\mathbf{M}_0\mathbf{x}_{0j} + \delta\mathbf{x}_i^H\mathbf{M}_0\mathbf{x}_{0j} + \mathbf{x}_{0i}^H\mathbf{M}_0\delta\mathbf{x}_j + \mathbf{x}_{0i}^H\delta\mathbf{M}\mathbf{x}_{0j}$$

To enforce $\mathbf{x}_i^H\mathbf{M}\mathbf{x}_j = \mathbf{x}_{0i}^H\mathbf{M}_0\mathbf{x}_{0j} = 0$, it follows that the linearized perturbations are zero

$$\delta\mathbf{x}_i^H\mathbf{M}_0\mathbf{x}_{0j} + \mathbf{x}_{0i}^H\mathbf{M}_0\delta\mathbf{x}_j + \mathbf{x}_{0i}^H\delta\mathbf{M}\mathbf{x}_{0j} = 0. \tag{224}$$

Next, expanding the new eigensystem and neglecting all quadratic perturbation terms, we arrive at

$$\mathbf{K}_0\delta\mathbf{x}_i + \delta\mathbf{K}\mathbf{x}_{0i} = \lambda_{0i}\mathbf{M}_0\delta\mathbf{x}_i + \lambda_{0i}\delta\mathbf{M}\mathbf{x}_{0i} + \delta\lambda_i\mathbf{M}_0\mathbf{x}_{0i} \tag{225}$$

Taking the inner product with $\mathbf{x}_{0i}$, taking the complex conjugate, and eliminating terms according to the orthogonality condition,

$$\begin{aligned}\mathbf{x}_{0i}^H\mathbf{K}_0\delta\mathbf{x}_i + \mathbf{x}_{0i}^H\delta\mathbf{K}\mathbf{x}_{0i} &= \lambda_{0i}\mathbf{x}_{0i}^H\mathbf{M}_0\delta\mathbf{x}_i + \lambda_{0i}\mathbf{x}_{0i}^H\delta\mathbf{M}\mathbf{x}_{0i} + \delta\lambda_i\mathbf{x}_{0i}^H\mathbf{M}_0\mathbf{x}_{0i} \\ \delta\mathbf{x}_i^H\left(\mathbf{K}_0\mathbf{x}_{0i} - \lambda_{0i}\mathbf{M}_0\mathbf{x}_{0i}\right) + \mathbf{x}_{0i}^H\delta\mathbf{K}\mathbf{x}_{0i} &= \lambda_{0i}\mathbf{x}_{0i}^H\delta\mathbf{M}\mathbf{x}_{0i} + \delta\lambda_i\end{aligned} \tag{226}$$

which yields

$$\delta\lambda_i = \mathbf{x}_{0i}^H \left(\delta\mathbf{K} - \lambda_{0i}\delta\mathbf{M}\right) \mathbf{x}_{0i}. \tag{227}$$

Note in the derivation above we used the fact that $\lambda_{0i} \in \mathbb{R}$ and $\delta\lambda_i \in \mathbb{R}$, when $\mathbf{K}_0$, $\mathbf{K}$ and $\mathbf{M}_0$, $\mathbf{M}$ are all Hermitian. The corrected eigenvalue thus takes the form

$$\lambda_i = \lambda_{0i} + \mathbf{x}_{0i}^H \left(\delta\mathbf{K} - \lambda_{0i}\delta\mathbf{M}\right) \mathbf{x}_{0i} = \lambda_{0i} + \frac{\mathbf{x}_{0i}^H \left(\delta\mathbf{K} - \lambda_{0i}\delta\mathbf{M}\right) \mathbf{x}_{0i}}{\mathbf{x}_{0i}^H \mathbf{M}_0 \mathbf{x}_{0i}}. \tag{228}$$

Secondly, in order to get the corrections to the eigenvectors, one needs to expand the perturbation in terms of unperturbed eigenvectors. This allows one to use the orthogonality and the completeness of the original eigenvectors.

$$\delta\mathbf{x}_i = \sum_j \varepsilon_{ij}\mathbf{x}_{0j}, \quad \varepsilon_{ij} = \frac{\mathbf{x}_{0j}^H \mathbf{M}_0 \delta\mathbf{x}_i}{\mathbf{x}_{0j}^H \mathbf{M}_0 \mathbf{x}_{0j}} = \mathbf{x}_{0j}^H \mathbf{M}_0 \delta\mathbf{x}_i \tag{229}$$

From the orthogonality condition eq.(224), One can conclude that the matrix elements ε_{ij} has the property:

$$\begin{aligned} \sum_k \overline{\varepsilon_{ik}} \mathbf{x}_{0k}^H \mathbf{M}_0 \mathbf{x}_{0j} + \sum_k \varepsilon_{jk} \mathbf{x}_{0i}^H \mathbf{M}_0 \mathbf{x}_{0k} + \mathbf{x}_{0i}\delta\mathbf{M}\mathbf{x}_{0j} &= 0 \\ \Longrightarrow \quad \overline{\varepsilon_{ij}} + \varepsilon_{ji} + \mathbf{x}_{0i}\delta\mathbf{M}\mathbf{x}_{0j} &= 0. \end{aligned} \tag{230}$$

The linearized eigensystem, on the other hand, can be used by taking the inner product with $\mathbf{x}_{0j}$, and taking the complex conjugate,

$$\begin{aligned} \sum_j \left(\mathbf{K}_0 - \lambda_{0i}\mathbf{M}_0\right) \varepsilon_{ij}\mathbf{x}_{0j} &= -\delta\mathbf{K}\mathbf{x}_{0i} + \lambda_{0i}\delta\mathbf{M}\mathbf{x}_{0i} + \delta\lambda_i\mathbf{M}_0\mathbf{x}_{0i} \\ \sum_j \left(\lambda_{0j} - \lambda_{0i}\right) \mathbf{x}_{0k}^H \mathbf{M}_0 \mathbf{x}_{0j} \varepsilon_{ij} &= -\mathbf{x}_{0k}^H \delta\mathbf{K}\mathbf{x}_{0i} + \lambda_{0i}\mathbf{x}_{0k}^H \delta\mathbf{M}\mathbf{x}_{0i} + \delta\lambda_i \mathbf{x}_{0k}^H \mathbf{M}_0 \mathbf{x}_{0i} \\ \left(\lambda_{0k} - \lambda_{0i}\right) \varepsilon_{ik} &= -\mathbf{x}_{0k}^H \left(\delta\mathbf{K} - \lambda_{0i}\mathbf{M}_0\right) \mathbf{x}_{0i} + \delta\lambda_i\delta_{ik}. \end{aligned} \tag{231}$$

For non-degenerate eigenvalues, as long as $i \neq k$, $\lambda_{0k} \neq \lambda_{0i}$. In this case, we directly obtain the solution

$$\varepsilon_{ik} = -\frac{\mathbf{x}_{0k}^H \left(\delta\mathbf{K} - \lambda_{0i}\mathbf{M}_0\right) \mathbf{x}_{0i}}{\lambda_{0k} - \lambda_{0i}}, \qquad \lambda_{0k} \neq \lambda_{0i}. \tag{232}$$

Problems arise when $i = k$. The modified eigensystem places no constraints on ε_{ii}. If both systems are in the real domain, one can conclude from the linearized orthogonality condition that $\varepsilon_{ii} = -\mathbf{x}_{0i}\delta\mathbf{M}\mathbf{x}_{0i}/2$; if systems are in the complex domain, however, one can only say $\mathrm{Re}[\varepsilon_{ii}] = -\mathbf{x}_{0i}\delta\mathbf{M}\mathbf{x}_{0i}/2$. The imaginary part of ε_{ii} only induces second-order effects in the magnitudes, and are thus absent from the linearized form.

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